Painleve III and a Matrix Model with Singular Weight

Lorna Brightmore, Francesco Mezzadri and Man Mo: arXiv:1003.2964
University of Bristol

An average over the GUE - Introduction and Motivation

We want to find the following average over the Gaussian Unitary Ensemble (GUE):

$$E_N(z;t) := \frac{1}{N!}\int_{\mathbb{R}^N}\prod_{j=1}^{N}\exp\left(-\frac{z^2}{2x_j^2}+\frac{t}{x_j}-\frac{x_j^2}{2}\right)\prod_{1\le j<k\le N}|x_k-x_j|^2 d^N x \tag{1}$$

Integrals such as this are known as **Partition Functions** and are extensively studied as they contain information on correlations of eigenvalues. This particular integral is of interest because certain statistics of eigenvalues of the GUE which are related to the study of zeros of the Riemann zeta function are encoded in the power series in t of $E_N(z;t)$.

Mezzadri, Mo (2009): For $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$,

$$E_N(z;t) = B_N \exp\left(-\frac{z^2}{4}-\frac{9}{2^{\frac{10}{3}}}N^{\frac{2}{3}}z^{\frac{4}{3}}\left(1+\frac{t^2N^{\frac{1}{3}}}{2^{\frac{5}{3}}z^{\frac{4}{3}}}\right)\right)(1+o(1)) \quad N\to\infty \qquad B_N := \int_{\mathbb{R}^N}\prod_{j=1}^{N} e^{-\frac{1}{2Nx_j^2}} P_{GUE}(x_1,\dots,x_N)\,d^N x \tag{2}$$

In the double scaling limit $z \; N^{-1}$