# Products with truncated unitary matrices

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## Introduction

- Recently, it became clear that in some cases the squared singular values of a product of random matrices still had a determinantal structure. For example, in the case of a product of Ginibre matrices, it was shown in [4, 3] that the squared singular values are a determinantal point process with joint p.d.f. proportional to
$$\prod_{j<k}(y_k - y_j) \det \left( g_{k-1}(y_j) \right)_{j,k=1}^{n}$$
where $g_k(y)$ is a Meijer G-function
- This was put in a more general framework showing that if a random matrix $X$ had a particular determinantal structure the product matrix $GX$, with $G$ a Ginibre random matrix, had the same structure [5]
- Two key ingredients for this proof:
  - Explicit formula for the distribution of $G$, namely $e^{-\mathrm{Tr}(G G)} dG$
  - Harish-Chandra/Itzykson-Zuber integral formula
- **Question**: could we replace $G$ by another random matrix such that the structure would still be preserved?

## The random matrix model

- Random matrix $X$ of size $l \times n$, with $l \geq n$
- Squared singular values of $X$ have j.p.d.f.
$$\prod_{j<k}(x_k - x_j) \det \left( f_k(x_j) \right)_{j,k=1}^{n}$$
- $U$ a Haar distributed random unitary matrix of size $m \times m$
- $T$ the $(n+\nu) \times l$ upper left submatrix of $U$

## Main result

Let $X$ and $T$ be as above. Then the squared singular values of $Y := TX$ have j.p.d.f.
$$\prod_{j<k}(y_k - y_j) \det \left( g_k(y_j) \right)_{j,k=1}^{n}$$
where
$$g_k(y) = \int_0^1 x^{\nu}(1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$
which is the Mellin convolution of $f_k$ with a beta distribution.

## Proof: First approach

For this approach we have to assume $m \geq 2n + \nu$. In this case there is an explicit formula for the distribution of a truncation of size $(n+\nu) \times n$ which we can use.

0. We may restrict to the case $l = n$.

Keep $X$ fixed:

1. Make the change of variables $T \to Y = TX$
2. Make the change of variables to the singular value decomposition $Y = U\Sigma V$
3. Integrate $U$ and $V$ over the unitary group. HCIZ-analogue integral formula:

## Integral over unitary group

Let $A$ and $B$ be $n \times n$ Hermitian matrices with respective eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$. Let $dU$ be the normalized Haar measure on the unitary group $U(n)$. Then for every $p \geq 0$,
$$\int_{U(n)} \det(A - UBU)^p \, \theta(A - UBU) \, dU$$