The Riemann Hilbert

- **One matrix model** = the asymptotic anal

The Riemann-Hilbert Method

- Solton Theory

Some of the Recent

tions

- Generalized Fisher-Hartwig Asymptotics

Krasovsky; Krasovsky, I; Deift, Krasovsky, I

- Asymptotics of Generalized Classical Orthogonal Polynomials

Toeplitz Determinants

Let $\phi(z)$ be a function defined on the unit circle,

$$C = \{z : |z| = 1\}.$$

The Toeplitz determinant, $D_n[\phi]$, is defined as

$$D_n[\phi] := \det T_n[\phi],$$

where

$$T_n[\phi] := \{\phi_{j-k}\}, \quad j, k = 0, \dots, n-1.$$

More generally,

$$\phi(z) \equiv \phi(z|\mathbf{t}),$$

$$D_n \equiv D_n(\mathbf{t}) \sim ?,$$

$$n, \mathbf{t} \to \infty.$$

- Multiple integrals and OPUC.

Classical Facts

- Strong Szegö theorem (1958)

$$D_n[\phi] \sim E[\phi](L[\phi])^n, \quad n \to \infty,$$

$$L[\phi] = \exp(\ln \phi)_0,$$

$$E[\phi] = \exp \sum_{k=1}^{\infty} k(\ln \phi)_k (\ln \phi)_{-k}$$

- Fisher-Hartwig asymptotics (1968)

$$\phi(z) = \phi_S(z) \prod_{r=1}^{N} |z - z_r|^{2\alpha_r}, \quad \alpha_r > -\frac{1}{2},$$

$$D_n \sim E[\phi_S](L[\phi_S])^n n^{\sum_r \alpha_r^2}$$

$$\times \prod_{r=1}^{N} \left(\frac{L(\phi_S)}{\phi_S(z_r)}\right)^{\alpha_r} \frac{G^2(\alpha_r + 1)}{G(2\alpha_r + 1)} \prod_{r \neq s} |z_r - z_s|^{-\alpha_r \alpha_s}.$$

$G(x)$ - Barnes' G-function.

- Generalization for the case of $\phi(z)$ with jumps

[illegible] Böttcher–Silbermann (1978–1985)

- Widom's theorem (1971)

$$\phi(z) = \phi_S(z)\chi_{C_\alpha}(z), \quad \phi_S(z) = \phi_S(\bar{z}),$$

$$C_\alpha : \alpha < \arg z < 2\pi - \alpha.$$

Here,

$$\psi_S(e^{i\theta}) = \phi_S\left(e^{2i \arccos(\gamma \cos\frac{\theta}{2})}\right).$$

Important particular case, $\phi_S(z) \equiv 1$:

$$D_n \sim \left(\cos\frac{\alpha}{2}\right)^{n^2} \left(n \sin\frac{\alpha}{2}\right)^{-1/4} e^{c_0},$$

• Widom's theorem for block Toeplitz matrices (1974)

$$\phi(z) - m \times m, \quad m \geq 1$$

$$D_n[\phi] \sim E[\phi](L[\phi])^n, \quad n \to \infty,$$

$$L[\phi] = \exp(\ln \det \phi)_0,$$

$$E[\phi] = \det\left(T_\infty[\phi] T_\infty[\phi^{-1}]\right),$$

$$\phi(z) - \text{“}good\text{''}.$$

- Further generalizations of the Fisher-Hartwig asymptotics (Krasovsky)

$$C_K = \left(\sin\frac{\alpha}{2}\right)^{-1/4}\left(\cos\frac{\alpha}{2}\right)^{-(2\alpha_1+\sum_r \alpha_r^2)}$$

$$\times 2^{4\alpha_1+2\alpha_1^2}\Gamma(1+2\alpha_1)\,G^2(3/2+2\alpha_1)$$

[illegible]

- The Fredholm determinant representation

$$D_n[\phi] = \det(I - K_n),$$

$$K_n : L_2(C; \mathbf{C}) \to L_2(C; \mathbf{C}),$$

$$K_n(z, z') = \frac{z^n (z')^{-n} - 1}{z - z'} \frac{1 - \phi(z')}{2\pi i},$$

(Deift)

Principal observation:

$$K_n(z, z') = \frac{f^T(z)h(z')}{z - z'},$$

$$f(z) = \begin{pmatrix} z^n \\ 1 \end{pmatrix},$$

[redacted]

• The Riemann-Hilbert representation

$$R := (I - K_n)^{-1} K_n,$$

$$R(z, z') = \frac{F^T(z) H(z')}{z - z'},$$

where

$$F(z) = Y_+(z) f(z), \quad H(z) = (Y_+^T)^{-1}(z) h(z),$$

$$z \in C,$$

1. $Y(z) \in H(\mathbf{C} \setminus C)$

2. $Y(\infty) = I_2$

3. $Y_-(z) = Y_+(z)G(z), \quad z \in C$

$$G(z) = I_2 + 2\pi i f(z) h^T(z)$$

$$= \begin{pmatrix} 2 - \phi(z) & -z^n(1 - \phi(z)) \\ z^{-n}(1 - \phi(z)) & \phi(z) \end{pmatrix}$$

Remark 1: [illegible]

$$\frac{D_{n+1}}{D_n} = Y_{11}(0)$$

Remark 2: Let [illegible]

$$\phi_t(z) := (1-t) + t\phi(z),$$

then

[illegible]

• An alternative RH. Relation to the orthogonal polynomials on the circle.

Observe:

$$G(z) = \begin{pmatrix} 2-\phi(z) & -z^n(1-\phi(z)) \\ z^{-n}(1-\phi(z)) & \phi(z) \end{pmatrix}$$

$$= \begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-n} & 0 \\ -1 & z^n \end{pmatrix}$$

Put:

$$\tilde{Y}(z) = Y(z)\begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix}, \quad |z| < 1,$$

$$\tilde{Y}(z) = Y(z)\begin{pmatrix} z^n & 0 \\ 1 & z^{-n} \end{pmatrix}$$

$$= Y(z)\begin{pmatrix} 1 & 0 \\ \text{[illegible]} & \end{pmatrix} z^{n\sigma_3}, \quad |z| < 1$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. $\tilde{Y}(z) \in H(\mathbf{C} \setminus C)$

2. $\tilde{Y}_{-}(z) = \tilde{Y}_{+}(z)\tilde{G}(z), \quad z \in C$

$$\tilde{G}(z) = \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}$$

3. $\tilde{Y}(z) z^{-n\sigma_3} \to I_2, \quad z \to \infty$

REMARK:

$$\tilde{Y}(z) \to \sigma_3 \tilde{Y}(z) \sigma_3$$

OPUC:

$$\tilde{Y}(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\int_C p_n(s)s^{-n}\phi(s)\frac{ds}{s-z} \\ q_{n-1}(z) & \frac{1}{2\pi i}\int_C q_{n-1}(s)s^{-n}\phi(s)\frac{ds}{s-z} \end{pmatrix}$$

$$\int p_k(z)\overline{p_l(z)}\phi(z)\frac{dz}{z} = h_k\delta_{kl}, \quad p_n(z) = z^n + \dots,$$

$$q_n(z) = -\frac{2\pi}{h_n^*}\overline{p_n^*\left(\frac{1}{\bar{z}}\right)}z^n,$$

$p_n^*(z)$ - orthogonal with respect to $\overline{\phi(z)}$.

PRINCIPAL IDEA.

Given $\phi(z)$, find $Y(z)$, such that

1. $Y(z) \in H\left(\mathbf{C} \setminus C\right)$

[illegible]

$$G(z) = \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}$$

3. $Y(z)z^{-n\sigma_3} \to I_2, \quad z \to \infty$

Then

$$Y_{11}(z) = p_n(z),$$

and

$$Y_{12}(0) = \frac{1}{\ }h_n \equiv \frac{D_{n+1}}{\ },$$

- Multiple integrals and OPRL.

$$D_n[\phi] = \frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j<k\le n}(z_j - z_k)^2$$

$$\prod_{1\le j\le n}\phi(z_j)dz_1\ldots dz_n.$$

$$\frac{D_{n+1}}{D_n} = h_n,$$

[illegible]

$$p_n(z) = z^n + \ldots,$$

Szegö type asymptotics

$$\left\{ \int_{-1}^{1} z^{k+j}\, \varphi(z)\, dz \right\}_{j,k=0,\dots,n-1}$$

$$\varphi(z) = (1-z)^{\alpha}(1+z)^{\beta}\,\omega(z)$$

$$\alpha, \beta > -1$$

$$h_n(\omega) \simeq \exp\Big\{ \frac{1}{\ } \int^{1} \ln \omega(z) \ldots \Big\}$$

Szegö type asymptotics

for infinite interval

[redacted]

$$\left\{ \int^{\infty} z^{j+k} \varphi(z)\,dz \right\}$$

[redacted]

Szegö type asymptotics

"Weak Szegö Theorem"

Equilibrium measure

"Strong Szegö Theorem"

$$V(z) = \frac{1}{2} z^2 + \sum_{k=1}^{2m} t_k z^k$$

one interval case

$$\ln \frac{D_n(t)}{D_n(0)} \simeq n^2 F_0(t) + F_1(t) + \frac{1}{n^2} F_2(t) + \dots$$

D. Bessis, C. Itzykson, J. B. Zuber

- Double scaling limit: $e^{-nV(z)}$

$V(z\,t) = \frac{1}{\cdot}\, z^4 + (1 - \frac{2}{\cdot})\, z^2$

two intervals

Root type singularity.

$$\left\{ \int^{\infty} z^{j+k} \varphi(z)\, dz \right\}$$

Jump

$$\varphi(z) = e^{-2nz^2} \begin{cases} e^{i\beta\pi} & z < \lambda \\ e^{-i\beta\pi} & z > \lambda \end{cases}$$

$$\lambda \in (-1,1) \qquad \beta \notin \mathbb{Z}$$

$D_n(\beta)$

$$C(\beta)\,(1-\lambda^2)^{-3\beta^2/2}\,(8n)^{-\beta^2}$$

Proof.

$$\frac{d}{\,} \ln D(R) = \int \ldots D(\lambda)\, D(\lambda) \ldots$$

$$\beta_n = \frac{h_n}{h_{n-1}}$$

$$\alpha_n = (2n)^{\frac{n-1}{2}} P_n^2(\lambda)\, e^{-\lambda^2} \operatorname{sh} \pi \gamma$$

- The RH problem.

Put:

$$Y(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} p_n(s)\phi(s)\frac{ds}{s-z} \\ q_{n-1}(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} q_{n-1}(s)\phi(s)\frac{ds}{s-z} \end{pmatrix}$$

$$q\,(z) = \frac{2\pi i}{\quad} p\,(z)$$

Then,

1. $Y(z) \in H(\mathbf{C} \setminus \mathbf{R})$

2. $Y_+(z) = Y_-(z)G(z), \quad z \in \mathbf{R}$

3. $Y(z)z^{-n\sigma_3} \to I_2, \quad z \to \infty,$

THE PRINCIPAL IDEA:

$$p_n(z) = Y_{11}(z),$$

and

$$h_n = \frac{i}{2\pi} \lim_{z\to\infty} z^{n+1} Y_{12}(z)$$

(**Fokas, Kitaev, I**, 1990)

Asymptotics. The NSD method

$$Y(z) \to \ldots \to Y_0(z):$$

$$||G_0(.) - I||_{L_2 \cap L_\infty} \leq \epsilon_n,$$

$$\epsilon_n \to 0, \quad \text{as} \quad n \to \infty.$$

Typically:

$$\epsilon_n = O\left(\frac{\ln^{\delta_1}}{n^{\delta_2}} e^{-cn^{\delta_3}}\right),$$

(Manakov (1973) — Deift&Zhou (1992))

Asymptotics of Toeplitz Determinants. A simple case

(**Deift; Jin, Korepin, I**) Assume that $\phi(z)$ is analytic in a neighborhood of the circle C.

Put

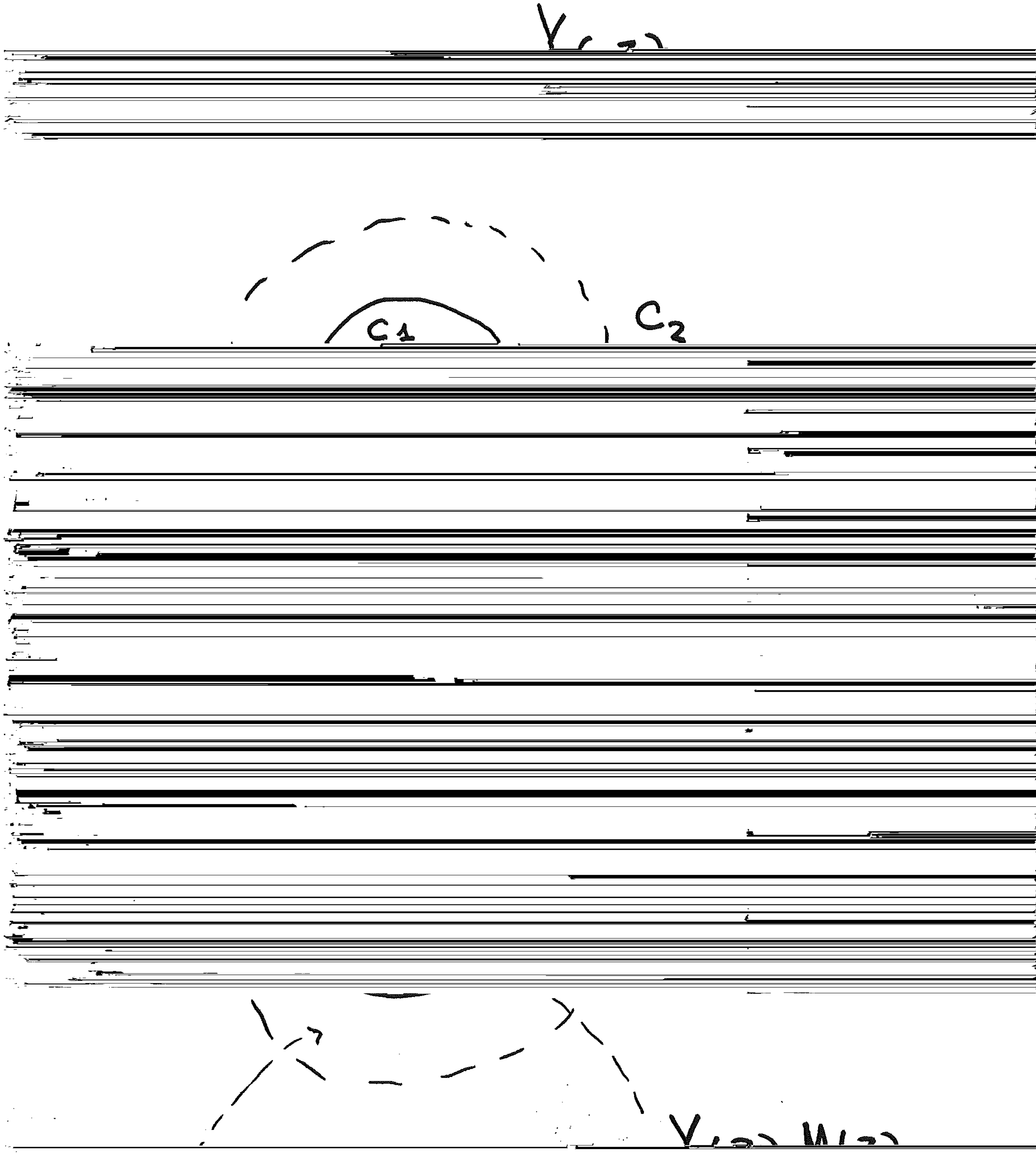
$$X(z) = Y(z)M(z) \quad \text{if} \quad 1 - \epsilon < |z| < 1,$$

$$X(z) = Y(z)N^{-1}(z) \quad \text{if} \quad 1 < |z| < 1 + \epsilon.$$

Denote

Put

$X(z)$:



$X(z)$ solves the following RH problem,

1^0. $X(z)$ is analytic outside of the contour [illegible]

[illegible]

$$\Gamma \equiv C \cup C_1 \cup C_2$$

2^0. $X(z)$ [illegible]

[illegible]

3^0. The jump of the function $X(z)$ across [illegible]

[illegible]

since

$$M(z), N(z) = I_2 + o(1) \quad n \to \infty,$$

we conclude that

$$X(z) \sim X^0(z)$$

where

$$X^0(z) = \begin{pmatrix} u_+(z) & 0 \\ 0 & u_+^{-1}(z) \end{pmatrix}, \quad \text{if} \quad |z| < 1,$$

and

$$X^0(z) = \begin{pmatrix} u_-^{-1}(z) & 0 \\ 0 & u_-(z) \end{pmatrix}, \quad \text{if} \quad |z| > 1.$$

Here:

$$\phi(z) = u_+(z)u_-(z)$$

- the Weiner-Hopf factorization of the symbol $\phi(z)$. From the above analysis, the classical Szegö and Widom's theorems follow.

The Widom-Dyson constant The Sine

where

[illegible]

$$D_n \sim \left(\cos\frac{\alpha}{2}\right)^{n^2}\left(n\sin\frac{\alpha}{2}\right)^{-1/4}$$

$$\times\, e^{[illegible]}, \quad n \to \infty$$

Observation

where

$$K_{\text{sine}} : L_2(0, 2s) \to L_2(0, 2s),$$

$$\sin \pi(x - y)$$

$$D_n = \left(\cos\frac{\alpha}{2}\right)^{n^2} \left(n \sin \alpha/2\right)^{-1/4}$$

$$\times\, e^{c_0} \left(1 + O\left(\frac{1}{n \sin \alpha}\right)\right)$$

(Deift, Krasovsky, Zhou, I)

$$\begin{pmatrix} 2 & -z^n \\ z^{-n} & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

g - **function:**

$$g(z) = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2z}$$

[redacted]

- $g(\infty) = 1$

- $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$

- $g_+(z) g_-(z) = \frac{\kappa}{z}, \quad \kappa = \cos^2 \frac{\alpha}{2}$

Put

$$X(z) = \kappa^{-\frac{n}{2}\sigma_3} Y(z) (g(z))^{-n\sigma_3} \kappa^{\frac{n}{2}\sigma_3}$$

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

[redacted]

- $X(\infty) = I$

- $X_-(z) = X_+(z) \begin{pmatrix} 2\left(\frac{g_+(z)}{g_-(z)}\right)^n & -1 \\ 1 & 0 \end{pmatrix}$

$z \in \Gamma_\alpha$

Since $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$, we expect that

$$X(z) \sim X^0(z) :$$

- $X^0(z)$ is holomorphic for all $z \notin \Gamma_\alpha$
- $X^0(\infty) = I$
- $X^0_-(z) = X^0_+(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  $z \in \Gamma_\alpha$

SOLUTION to the MODEL PROBLEM:

$$X^0(z) = \begin{pmatrix} \frac{\delta+\delta^{-1}}{2} & \frac{\delta-\delta^{-1}}{2i} \\ -\frac{\delta-\delta^{-1}}{2i} & \frac{\delta+\delta^{-1}}{2} \end{pmatrix},$$

where

[illegible]

More careful analysis (see Appendix) along the lines indicated allows to obtain the following (rigorous) extension of (2).

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} \Delta(n, \alpha), \qquad (3)$$

Multiple integral:

$$D_n(\alpha)$$

$$\int^{2\pi-\alpha} \int^{2\pi-\alpha} \cdots$$

• From the multiple integral representation of $D_n(\alpha)$ one obtains

$$\ln D_n(\alpha) = n^2 \ln(\pi - \alpha) - n \ln 2\pi$$

$$+ \ln A_n + O((\pi - \alpha)^2), \qquad (4)$$

where

$$A_n = \prod^{n-1} \frac{2^{2k}(k!)^4}{\quad} \frac{2}{\quad}$$

REMARK:

$$\det(1 - K_{\text{sine}}) = \exp\left(\int_0^s \frac{\sigma_V(t)}{t} dt\right)$$

where $\sigma_V(s)$ - the special solution of P_V:

$$(s\sigma'')^2 + 4(4\sigma - s\sigma' - (\sigma')^2)(\sigma - s\sigma') = 0,$$

$$\sigma(s) \sim -\frac{2}{\pi}s, \quad s \to 0.$$

(**JMMS**)

The Dyson - Ehrhardt - Krasovsky formula implies that

$$v.p. \int_0^\infty \frac{\sigma_V(t)}{t} dt = 3\zeta'(-1) + \frac{1}{12}\ln 2$$

Appendix

Put

$$\Phi(\lambda) := X^{-1}(-1)X(z(\lambda)),$$

where

$$z(\lambda) = \frac{1 + i\lambda \tan \frac{\alpha}{2}}{1 - i\lambda \tan \frac{\alpha}{2}}$$

maps the interval $[-1, 1]$ to the arc Γ_α.

- $\Phi(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$

- $\Phi(\infty) = I$

Note:

- The Φ - problem is regular in the neighborhood of $\alpha = \pi$

- The following relation takes place

- D_n satisfies Painlevé VI equation:

$$\eta(t) \equiv t(t-1)\frac{d}{dt}\ln D_n, \qquad t \equiv e^{-2i\alpha}$$

$$\left(\frac{d\eta}{dt}\right)^4 = \left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t(t-1)\frac{d^2\eta}{dt^2}\right)^2$$

$$+\left[2\left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t\frac{d\eta}{dt} - \eta\right) - \left(\frac{d\eta}{dt}\right)^2 + \frac{n^2}{2}\frac{d\eta}{dt}\right]^2.$$

$$\Delta = \frac{1-t}{n^2}\frac{d\eta}{dt} + \frac{1}{n^2}\eta.$$

$$Y_-(z) = Y_+(z) \begin{pmatrix} 1 & e^{-2nz^2}\omega(z) \\ 0 & 1 \end{pmatrix}$$

$Y \ldots (I + O(\frac{1}{z}))\, z^{n\sigma_3}$ ...

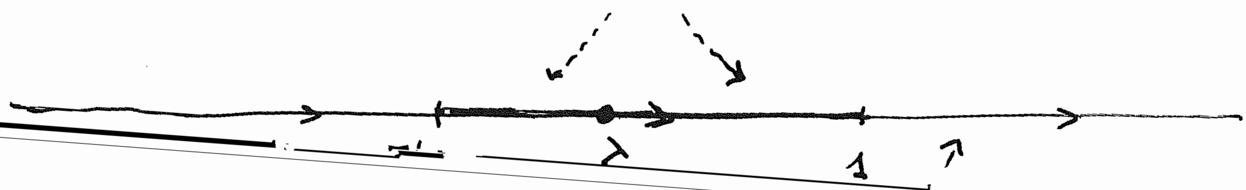
$$T(z) := e^{-n\ell\sigma_3/2}\, Y(z)\, e^{-n(g(z)-\ell/2)\sigma_3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $T(z) = I + O(1/z)\ , \quad z \to \infty$

- $T_+ = T_-\, G_T(z)\ :$

$$\begin{pmatrix} e^{-nh} & \omega \\ 0 & e^{nh} \end{pmatrix}$$



[illegible]

0 1

[illegible]

$$\begin{pmatrix} e^{-nh} & \omega \\ \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ \end{pmatrix} \begin{pmatrix} 0 & \omega \\ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \end{pmatrix}$$

[illegible]

$$S^{(\infty)}(z):$$

$$\begin{pmatrix} 0 & e^{-i\beta\pi} \\ -e^{+i\beta\pi} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & e^{+i\beta\pi} \\ -e^{-i\beta\pi} & 0 \end{pmatrix}$$

$S^{(\pm 1)}(z)$:

$$\begin{pmatrix} 1 & 0 \\ e^{\zeta^{3/2}} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ e^{+\zeta^{3/2}} & 1 \end{pmatrix}$$

$\zeta^{3/2}$:

ζ [illegible] $\left(\ldots \int^{z} \sqrt{\ldots} \ldots \right)^{3/2} = c_0 (z-1) + \ldots\ldots$

$S^{(\lambda_0)}(z)$: