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# Entanglement entropy in quantum spin chains with nite range interaction

Francesco Mezzadri

В

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Fundamentals and Applications
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Work in collaboration with Alexander Its and Man Yue Mo



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- 2 Spin chains and block-Toeplitz determinants
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#### Outline

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## The problem

Consider a one-dimensional quantum spin chain:

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## The problem

Consider a one-dimensional quantum spin chain:

Look at the ground state j gih gj

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## The problem

Consider a one-dimensional quantum spin chain:

Look at the ground state j  $_g$ ih  $_g$ j ( $\mathcal{T}=0$ : phase transition in the thermodynamic limit.)

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Look at the ground state j  $_g$ ih  $_g$ j ( $\mathcal{T}=0$ : phase transition in the thermodynamic limit.) Questions we can ask:



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## The problem

Consider a one-dimensional quantum spin chain:

Look at the ground state j gih gj (T = 0: phase transition in the thermodynamic limit.) Questions we can ask:

What is the entropy of the entanglement between A and B

as L! 1 ?

What is the correlation between two spins at di erent sites?

Many others.

If the model is:

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## The problem

Jin and Korepin (2004):

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### The problem

Jin and Korepin (2004): Spatial isotropy ( = 0), next neighbour interaction and translation invariance (XX model, Toeplitz determinants).

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## The problem

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Keating and Mezzadri (2004): Spatial isotropy ( = 0), nite range interaction, translation invariance and re ection symmetries ( ) averages over the classical compact groups (Toeplitz + Hankel determinants.)

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Its, Jin and Korepin (2006):

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Its, Jin and Korepin (2006): Spatial anisotropy ( € 0), next neighbour interaction and translation invariance. (XY model, block-Toeplitz determinants.)

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## Spin chains and block-Toeplitz determinants

The entropy of the entanglement can be written as

$$S(A) = \lim_{i \to 0^{+}} \lim_{L \to 1} \frac{1}{4i} \Big|_{(i)} e(1 + i) \Big|_{(i)}$$

$$\frac{d}{d} \log D_{L}(i) \Big(i^{2} - 1\big)^{L} d = i$$

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# Spin chains and block-Toeplitz determinants

 $D_L(\ )$  is the determinant of the block-Toeplitz matrix

$$T_{L}[\ ] = \begin{bmatrix} O & & & & & & 1 \\ B_{0} & B_{1} & & B_{2} & L & B_{1} & L \\ B_{1} & B_{0} & & B_{3} & L & B_{2} & L \\ B_{1} & B_{1} & B_{1} & B_{2} & B_{1} & B_{0} \\ B_{L} & 2 & B_{L} & 3 & B_{0} & B_{1} & A \\ B_{L} & 1 & B_{L} & 2 & B_{1} & B_{0} \end{bmatrix}$$

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## Spin chains and block-Toeplitz determinants

$$g^2(z) = \begin{cases} Y^n & z & z_j \\ j=1 \end{cases}$$

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## Spin chains and block-Toeplitz determinants

$$g^2(z) =$$







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## Spin chains and block-Toeplitz determinants

### Remarks:

The branch cuts of g(z) are the segments

$$i = \begin{bmatrix} 2i & i \\ 2i \end{bmatrix}; \quad i = 1 \\ \vdots \\ 2n$$

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## Spin chains and block-Toeplitz determinants

### Remarks:

The branch cuts of g(z) are the segments

$$i = [2i \ i; 2i]; i = 1; :: : ; 2n$$

$$g(z)$$
 has discontinuities  $g_+(z) = g(z)$ ;  $z = 2$ 

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## Spin chains and block-Toeplitz determinants

### Remarks:

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$$i = \begin{bmatrix} 2i & i \\ 2i \end{bmatrix}; \quad i = 1; :: : ; 2n$$

g(z) has discontinuities  $g_+(z) = g_-(z)$ ;  $z \ge i$ g(z) lives on the hyperelliptic curve

$$L: W^2 = \bigvee_{i=1}^{4n} (Z \qquad i):$$

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## Spin chains and block-Toeplitz determinants

### Remarks:

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$$L: W^2 = \bigvee_{i=1}^{4n} (Z \qquad i):$$

The genus of L is q = 2n 1.

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De ne : 
$$C^g$$
 !  $C$  associated to  $L$  by 
$$(\stackrel{!}{s}) := \underset{\stackrel{!}{n}}{\times} e^{i \stackrel{!}{n} - \stackrel{!}{n} - 2 \stackrel{!}{i} \stackrel{!}{s} \stackrel{!}{n}} :$$

is a g-g symmetric matrix (period matrix) that depends on the Hamiltonian through the branch cuts of L.

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What happens at a phase transition?

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# Block-Toeplitz determinants and the RH problem

How do we compute such formulae?



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## Block-Toeplitz determinants and the RH problem

How do we compute such formulae?

The symbol  $(z) = \int_{g^{-1}(z)}^{i} \frac{g(z)}{i}$  admits the Wiener-Hopf factorization:

$$(z) = U_{+}(z)U_{-}(z) = V_{-}(z)V_{+}(z);$$

where U(z) and V(z) are analytic inside/outside the unit circle and

$$U(1) = V(1) = I$$
:

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# Block-Toeplitz determinants and the RH problem

Theorem (Widom, 1974)

$$\frac{d}{d} \log D_L(\ ) = \frac{2}{1 - 2} L$$

$$+ \frac{1}{2} \int_{S^1}^{L} tr U_+^0$$

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# Block-Toeplitz determinants and the RH problem

It turns out that

$$V(z) = {}_{3}U(z) {}^{1}{}_{3}$$
  
 $V_{+}(z) = {}_{3}U_{+}(z) {}^{1}{}_{3}({}^{2}{}^{1});$   $\leftarrow$  1

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## Block-Toeplitz determinants and the RH problem

Let us de ne

$$S(z) = U(z)Q(z)^{-1};$$
 jzj 1;

$$S(z) = U_{+}(z)^{-1}Q(z);$$
 jzj 1:

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In order to compute the entropy of entanglement we need:

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In order to compute the entropy of entanglement we need:



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In order to compute the entropy of entanglement we need:

lacktriangledown to solve the previous RH problem for S(z) in terms of)

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# The critical case (phase transitions)

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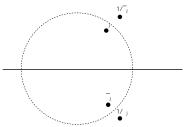
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## The critical case (phase transitions)

Pairs of roots of g(z) approach the unit circle.



The period matrix in

$$({}^{!}s) := \frac{X}{{}^{!}_{n \ 2Z^{g}}} e^{i \ {}^{!}_{n} \ {}^{!}_{n} \ 2i \ {}^{!}_{s} \ {}^{!}_{n}}.$$

becomes degenerate.

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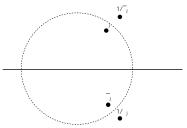
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Pairs of roots of g(z) approach the unit circle.



The period matrix in

$$(!s) := \frac{X}{!_{n \, 2Z^g}} e^{i \cdot !_n \cdot !_{n \, 2i} \cdot !_s \cdot !_n}$$

becomes degenerate.

(!s ) becomes singular.



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### Summary

We computed the entropy of the entanglement of the ground state of integrable quantum spin chains with nite range and translation invariant interaction.

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### Summary

We computed the entropy of the entanglement of the ground state of integrable quantum spin chains with nite range and translation invariant interaction.

At the core of the computation is the evaluation of block-Toeplitz determinants.

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### Summary

We computed the entropy of the entanglement of the ground state of integrable quantum spin chains with nite range and translation invariant interaction.

At the core of the computation is the evaluation of block-Toeplitz determinants.

Such determinants are computed by solving a RH problem.

At phase transition we observe logarithmic divergences that generalize previous results.

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AR Its, F Mezzadri and MY Mo. Entanglement entropy in quantum spin chains with nite range interaction. arXi v: 0708, 0161v1.