

Our activity in this direction was triggered by the paper by Dean and Majumdar, who came up with an approximate theory of large spectral fluctuations in the three Gaussian Random Matrix models – GOE, GUE and GSE.

# Non-perturbative Theory of Large Spectral Deviations in CUE

PHYSICAL REVIEW LETTERS 20 OCTOBER 2006 PRL 97, 160201 (2)

## Large Deviations of Extreme Eigenvalues of Random Matrices

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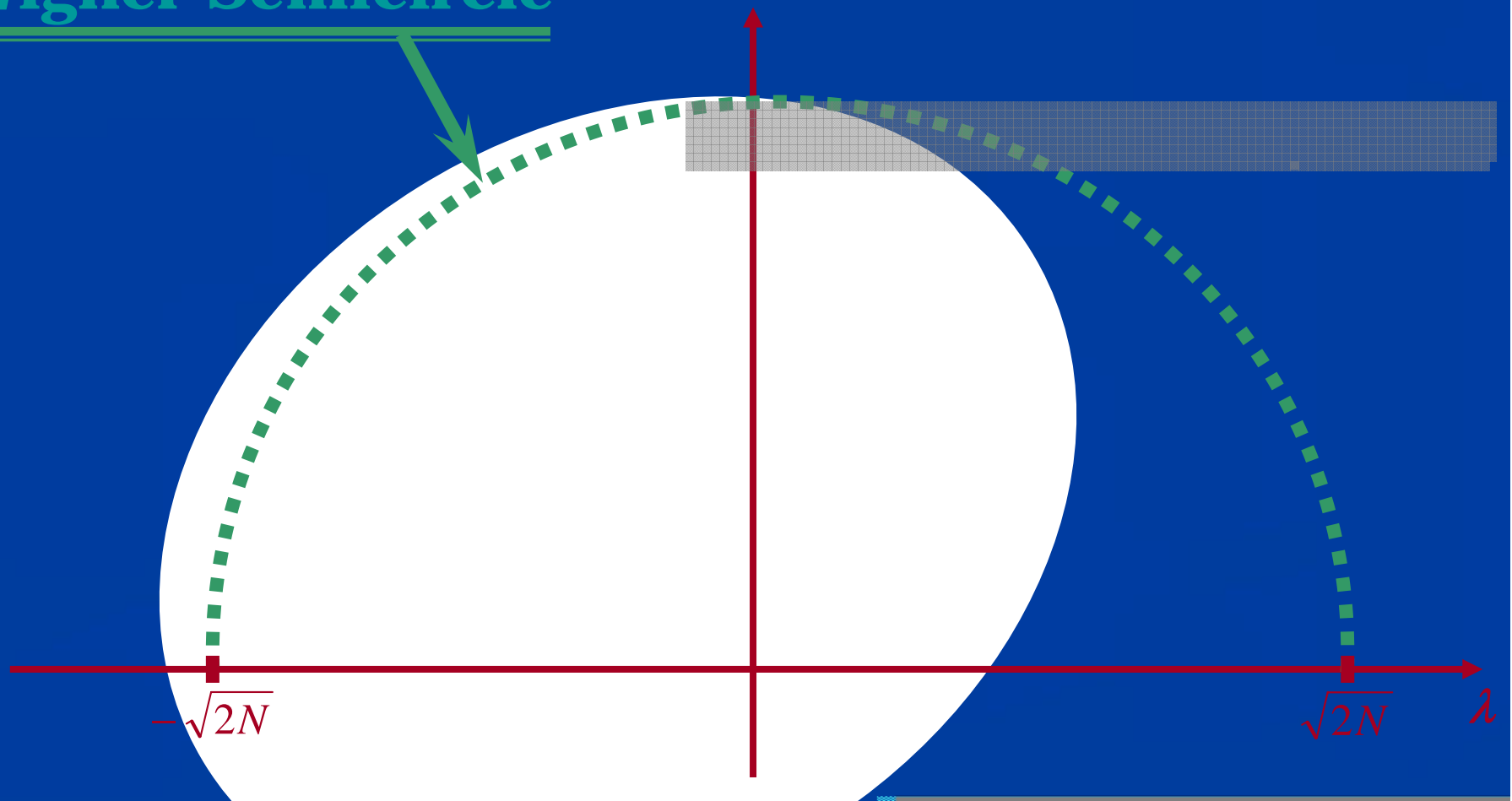
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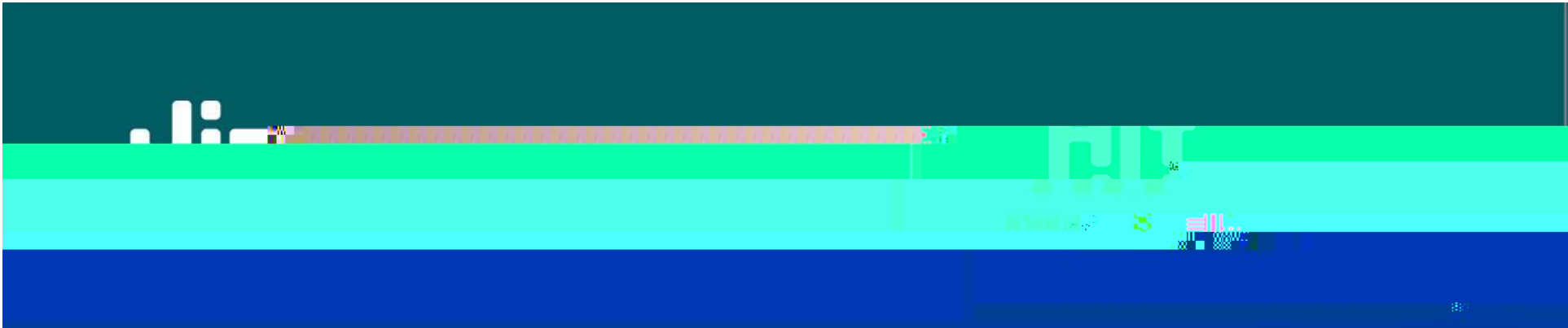
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(Received 3 August 2006; published 20 October 2006)

# Wigner Semicircle





-1 -0.5 0 0.5 1 1.5 2

# The Major Statement of the Work

We managed to find an

D.S. Dean & S.N. Majumdar result

$$D_{\text{UE}}(z) = \frac{1}{2\pi} \int_{-2}^2 \frac{1}{z - \lambda} \rho(\lambda) d\lambda + \mathcal{O}(N^{-1})$$

GUE<sub>NxN</sub> N >> 1

$$3^{1/8} \tau_0 S_0(z)$$

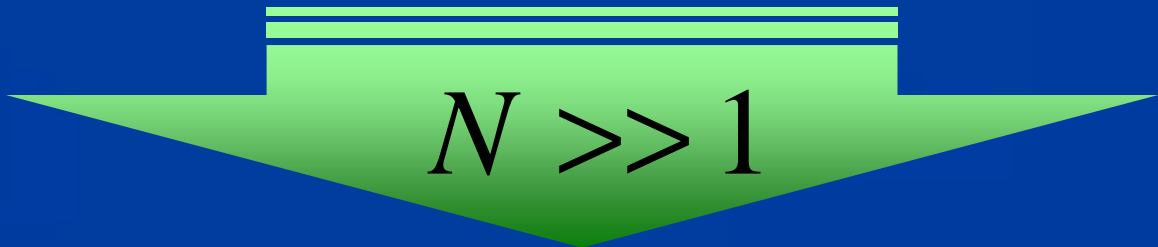
$$1 - \frac{1}{2} \frac{z^2 + 3}{z^2 + 3} + \frac{1}{2} \frac{z^2 + 3}{z^2 + 3}$$

$$\frac{(\sqrt{z^2 + 3} - z)^{1/6}}{(\sqrt{z^2 + 3} - z)^{1/8} (\sqrt{z^2 + 3})^{1/48}}$$

# Probability Distribution Function via Coulomb Fluid Approach

An absence of the linear in N term can be appreciated even within the Coulomb fluid approach. Indeed, for general beta, the desired distribution function  $P_N$  which is defined by a matrix integral can approximately be rewritten as a functional integral. This representation holds in the large N-limit and each term in the action has a clear origin.

$$d\mu_j P_N(z) \sim \int_{-\infty}^z |\Delta_N(\boldsymbol{\mu})|^\beta \prod_{j=1}^N e^{-\beta N \mu_j^2}$$



$$P_N(z) \sim \int \mathcal{D}[\rho] \exp \left[ -\beta N^2 \int_{-\infty}^z \mu^2 \rho(\mu) d\mu \right]$$

$$\beta N^2 \int_{-\infty}^z \mu^2 \rho(\mu) d\mu$$

$$\sum_{i=1}^N \delta(\mu - \mu_i) \rho(\mu) = \frac{1}{N}$$

$$\int_{-\infty}^z \mu^2 \rho(\mu) d\mu = \frac{1}{N} \int_{-\infty}^z \mu^2 d\mu$$

# Coulomb Fluid Approach

## F. Dyson (1962)

$$1. e^{-\beta N \sum_j \mu_j^2} \xrightarrow{N \gg 1} e^{-\beta N^2 \int^z u^2 \rho(u) du}$$

$$2. \frac{1}{N!} \int \prod_{j=1}^N d\mu_j e^{-\beta \sum_{j=1}^N \mu_j^2} e^{-\beta \sum_{j=1}^N \sum_{k=1}^N \mu_j \mu_k}$$

Short-distance regularization  
local level spacing  $1/(N\rho(u))$

$$\xrightarrow{N \gg 1} \frac{\beta}{2} \sum_{j=1}^N \ln|\mu_j - \mu_k| \rightarrow \frac{\beta}{2} \sum_{j=1}^N \ln \left| \frac{1}{N \rho(\mu)} \right|$$

$$\frac{\beta N^2}{2} \int_{-\infty}^z \rho(\mu) \rho(\mu) d\mu \rightarrow \frac{\beta N^2}{2} \int_{-\infty}^z \rho(\mu) d\mu$$

$$3. \int_{-\infty}^z \prod_{j=1}^N d\mu_j \xrightarrow{N \gg 1} \int_{-\infty}^z \rho(\mu) d\mu$$

$$\rho(\mu) = \frac{1}{N} \sum_{j=1}^N \delta\left(\mu - \frac{\lambda_j}{\sqrt{N}}\right)$$

$$\int_{-\infty}^z \rho(\mu) d\mu = 1$$

# Absence of the Linear in N Term. Coulomb Fluid Approach

At  $\beta=2$ , the linear in N term vanishes. This lead us to conclude that a linear in N term cannot be present in the Dean & Majumdar solution for  $\beta=2$ .

$$d\mu_j P_N(z) \sim \int_{-\infty}^z |\Delta_N(\boldsymbol{\mu})|^\beta \prod_{j=1}^N e^{-\beta N \mu_j^2}$$

$$N \gg 1$$

$$P_N(z) \sim \int \mathcal{D}[\rho] \exp \left[ -\beta N^2 \int_{-\infty}^z \mu^2 \rho(\mu) d\mu \right]$$

$$\beta N^2 \int_{-\infty}^z \mu^2 \rho(\mu) d\mu$$

$$\sum_{i=1}^N \delta(\mu - \mu_j) \rho(\mu) = \frac{1}{N}$$

~~$$\beta N^2 \int_{-\infty}^z \mu^2 \rho(\mu) d\mu$$~~

# Non-perturbative Theory of Large Spectral Deviations

Our consideration is based on the exact representation of the probability distribution function in terms of a solution of the fourth Painlevé transcendent. This formula can be derived by using a technique of Fredholm determinant (as was done by Tracy and Widom), or with the help of theory of integrable hierarchies much in line with the works by Adler and van Moerbeke.

A complex mathematical expression involving Fredholm determinants and integrals. It features terms like  $Nu^2$ ,  $D(\lambda)$ ,  $\prod_{j=1}^N$ , and  $\int_{-\infty}^{\infty} d\lambda f(\lambda)$ .

Integral representation of the Fredholm determinant  $D(z)$ . The expression is  $D(z) = \det(I - K_z)$ , where  $K_z$  is an integral operator with kernel  $k(x, y; z)$  and integration over  $x$  and  $y$  from  $-\infty$  to  $+\infty$ .

## Painlevé IV in Chazy form

Chazy form of the Painlevé IV equation:  $6w''^2 - 5w'w'' - w^2w'' + 4w'w'' - 4w^2w'' - 4w'w'' + 4w^2w'' = 0$

### 1. Fredholm determinant representation.

C.A. Tracy & H. Widom (1992)

### 2. Integrable hierarchies. M. Adler & P. van Moerbeke (2001)



# Non-perturbative Theory of Large Spectral Deviations in GUE

The appropriate boundary conditions were determined by Tracy and Widom who made us the result by Bassom Clarkson Hicks and McLeod.

$$D_N(x) \sim \int_{-\infty}^{+\infty} \dots$$

$$\dots$$

## Boundary conditions

A.P. Bassom, P.A. Clarkson, A.C. Hicks, J.B. McLeod  
(1992)

$$f_N(x) \Big|_{x \rightarrow +\infty} = 0$$

$$\dots$$

$$\dots + N^0 \left( \dots + \dots + 0 \right)$$

# 1/N Expansion

Existence of the large parameter N makes it tempting to seek the solution  $f_N$  in the form of the 1/N expansion. We may argue that Gamma is identically zero. The equations for Lambda and Omega together with appropriate boundary conditions follow directly from the initial equation for  $f_N$ .

$$f_N(x)|_{x \rightarrow \infty} = 0$$

$$\left( \frac{d}{dx} + \frac{2x}{32x^7} + \mathcal{O}(x^{-8}) \right) f_N(x) \Big|_{x \rightarrow -\infty} = N^2 \left( -4x - \frac{27}{x} + \frac{12}{2x^3} - \frac{1}{16x^7} \right) + N^0 \left( \frac{1}{x} + \frac{10}{16x^5} + \mathcal{O}(x^{-8}) \right)$$

## Ansatz

$$f_N(x) = N^2 \Lambda(x) + N \Gamma(x) + \Omega(x) + \mathcal{O}(N^{-1})$$

$$f_N(z) \simeq e$$

$$6(\Lambda'(x))^2 + 16(1-x^2)\Lambda'(x) + 16x\Lambda(x) = C$$

$$\Gamma'(x) = \dots$$

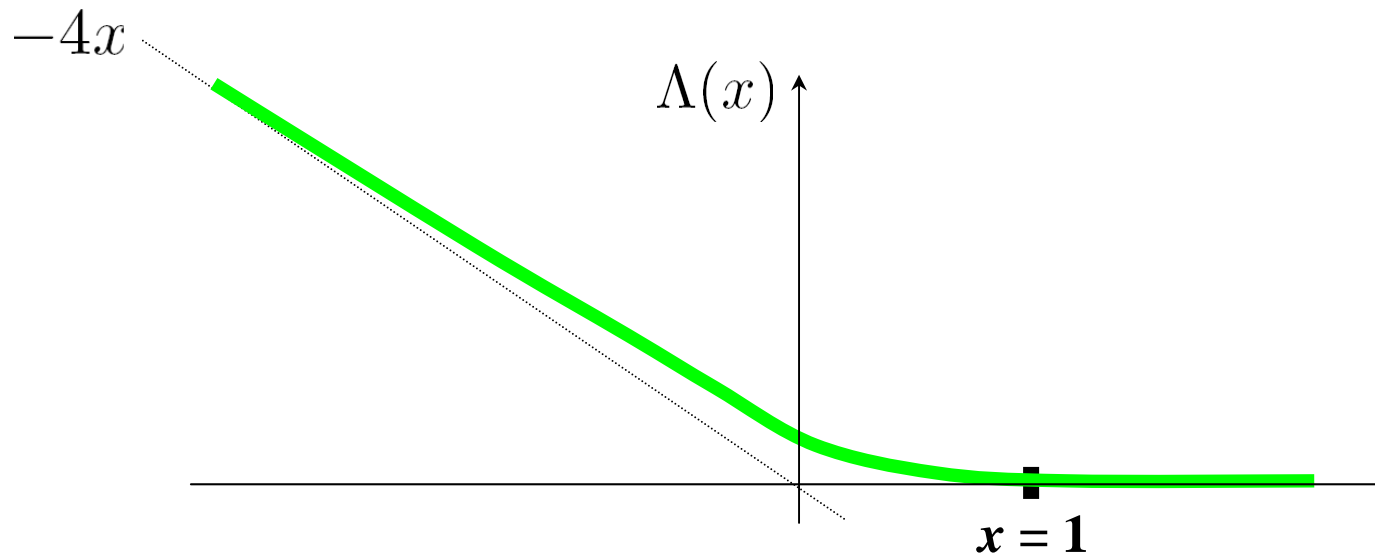
$$\Omega(x) = \dots$$

# $\Lambda(x)$

The explicit solution for Lambda, that has continuous derivative, is shown on the graph. At minus infinity it approaches the line  $-4x$ , as expected. At  $x$  large then one, Lambda vanishes.

$$6 (\Lambda'(x))^2 + 16(1 - x^2)\Lambda'(x) + 16x\Lambda(x) = C$$

$$\Lambda(x) \Big|_{x \rightarrow -\infty} = -4x \quad \Lambda(x) \Big|_{x \rightarrow \infty} = 0$$





# Non-perturbative Theory

$$x = 1 + \frac{s}{2N^{2/3}}$$

$$f(x) = \frac{1}{2N^{2/3}} f\left(1 + \frac{s}{2N^{2/3}}\right)$$

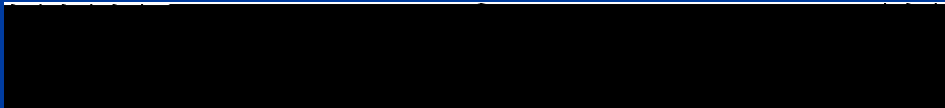
# Non-perturbative Theory of Large Spectral Deviations in GUE

## Matching with the Tracy-Widom Distribution

$$P_N(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z} \exp\left(-\int_z^{+\infty} \varphi(t) dt\right) dt$$

$$\int_{-cN^{-2/3}}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots$$

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \dots$$



$$F_2(s)$$

$$\approx \frac{2^{5/24}}{s^{1/8}} N^{1/12} e^{-|s|^3/12}$$



$$\approx \tau_0 \frac{e^{-|s|^3/12}}{|s|^{1/8}}$$

# Non-perturbative Theory of Large Spectral Deviations in GUE

## The Major Statement of the Work

**GUE<sub>N×N</sub>  $N \gg 1$   $z \in$  ( Large fluctuations domain )**

$$3^{1/8} \tau_0 S_0(z)$$

**$z=0, N = 10$**

Numerics :  $9.95926 \times 10^{-25}$

Dean & Majumdar:  $1.39296 \times 10^{-24}$

Exact solution:  $9.95935 \times 10^{-25}$

$$\delta = \left| \frac{\text{Numerics} - D \& M}{\text{Numerics}} \right| \sim \mathbf{40}$$




	1	2	5	10	20	50			
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## Conclusions

**Asymptotically exact solution for the probability of Large Deviations of  $\lambda_{\max}$  for  $\beta=2$**

**No linear in N term in the exponent at  $\beta=2$**

**Large fluctuations and Tracy Widom formula are two different scaling limits of the same theory**

**Similar approach can be developed for  $\beta=1$  & 4**