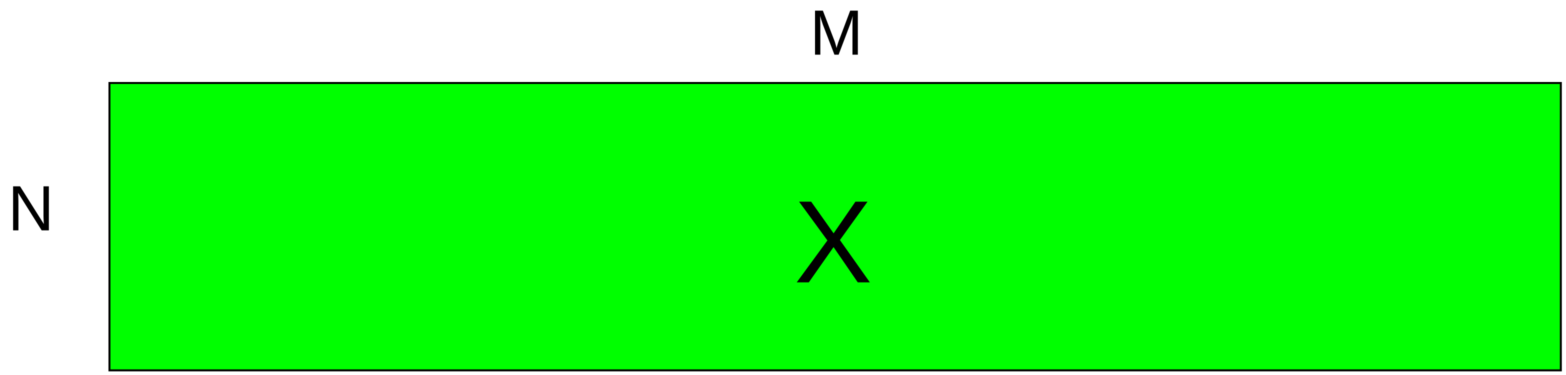
# Superstatistical Generalisations of Wishart-Laguerre ensembles

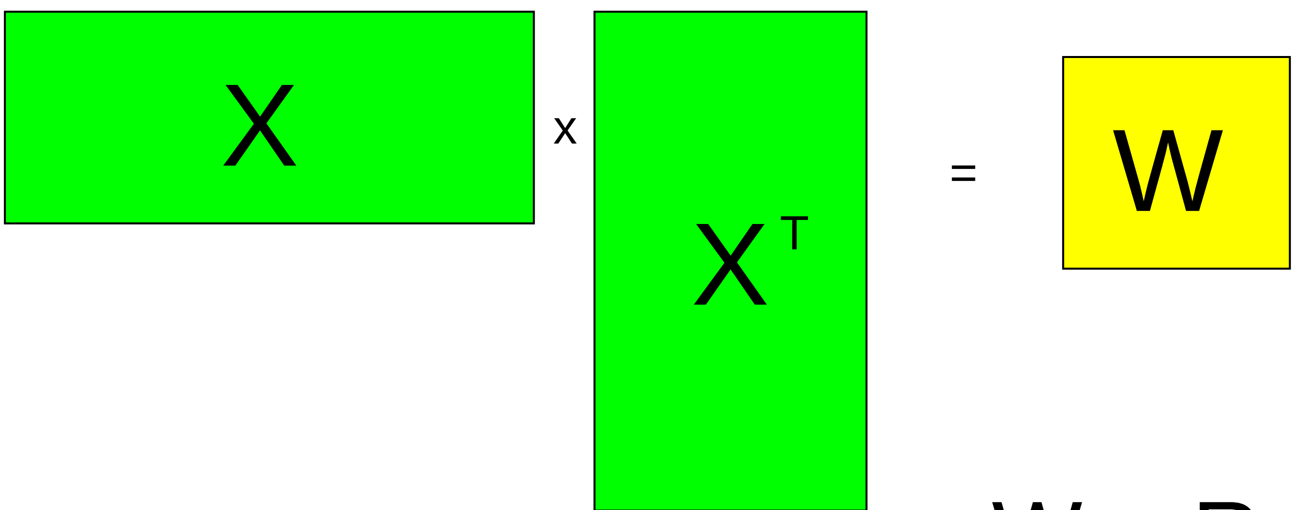
Pierpaolo Vivo
(ICTP - Trieste)

in collaboration with G. Akemann and A.Y. Abul-Magd

Brunel RMT Workshop
19th December 2008




$$\mathbf{X}_{ij} \sim \mathcal{N}(0,1)$$




W = Random Covariance Matrix

# Wishart-Laguerre ensemble

- Introduced by John Wishart in 1928
- JPD of eigenvalues (real and positive) is known

$$P(\lambda_1, \ldots, \lambda_N) = C_N \prod_{i=1}^{N} e^{-\frac{1}{2}\lambda_i} \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

- Density of eigenvalues for $N, M \to \infty$: $N/M = c$

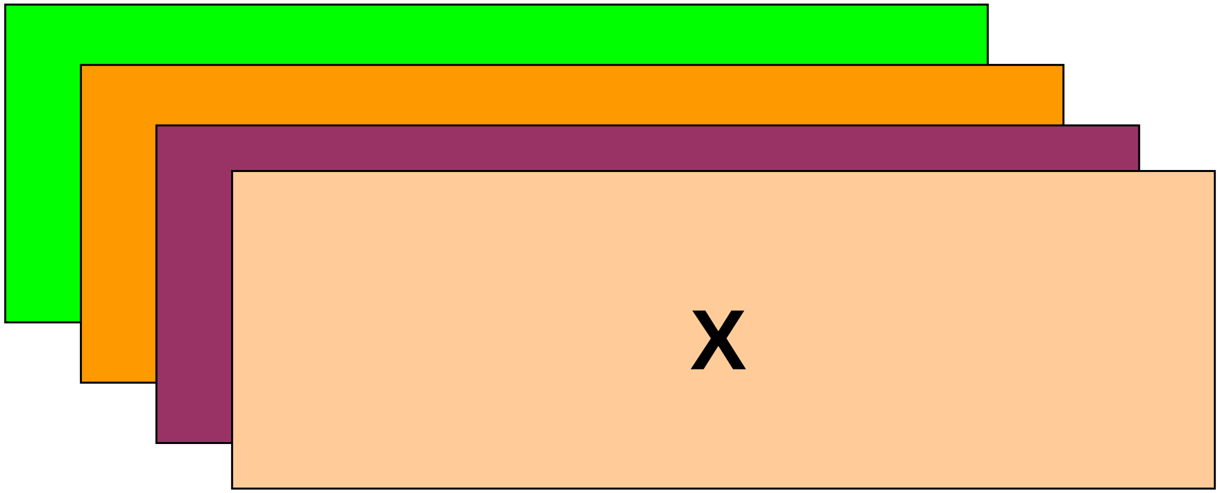
$$\rho(\lambda; c) = \beta N^{-1} f(\beta N^{-1} \lambda)$$

$$f(x) = \frac{1}{2\pi x}\sqrt{(x - X_-)(X_+ - x)}$$

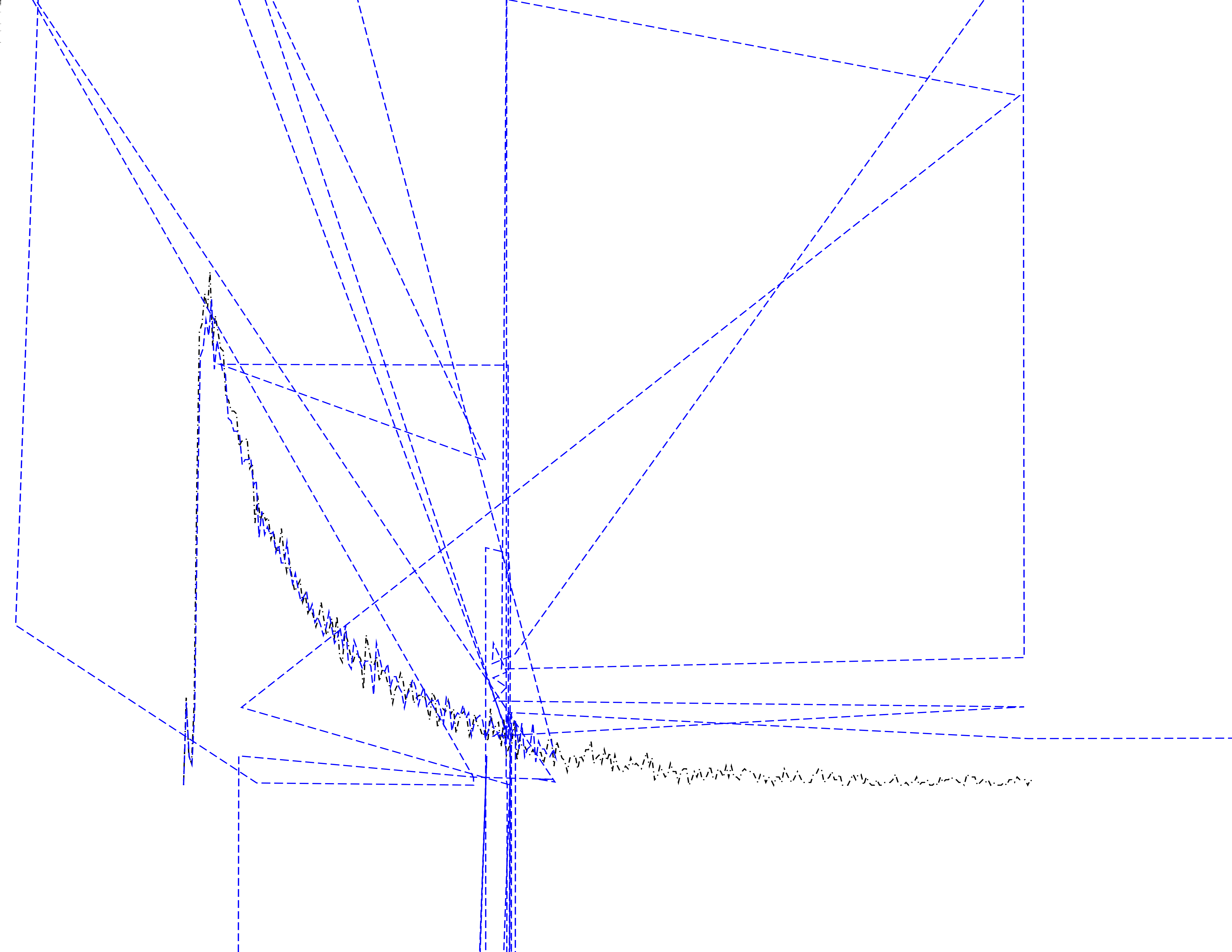
# Superstatistics

- Beck and Cohen (2003)
- Simple description of non-equilibrium

# Superstatistical model




- The variance of $\mathbf{X}$-entries fluctuates from one sample to another
- The spectral density and all correlation functions of $\mathbf{W} = \mathbf{X}\mathbf{X}^{\dagger}$ are modified
- The model is exactly solvable



# Three Superstatistical Classes

- $\chi^2$-distribution

[illegible]

- Inverse $\chi^2$-distribution

$$f_2(\eta) \propto \frac{1}{[illegible]} \exp\left(-\gamma/\eta\right)$$

- Log-Normal Distribution

[illegible]

$$P_2\left(\mathbf{X}\right) \longrightarrow \exp\left[-\eta\beta \mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]$$

$$P_2\left(\mathbf{X}\right) \longrightarrow \frac{1}{\eta^{\gamma+2}} \exp\left[-\frac{\gamma}{\eta}\right] \exp\left[-\eta\beta \mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]$$

$$P_2\left(\mathbf{X}\right) \longrightarrow \int_0^\infty d\xi \frac{1}{\xi^{\gamma+2}} \exp\left(-\frac{1}{\xi}\right) \exp\left[-\xi\beta\gamma \mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]$$

$$P_2\left(\mathbf{X}\right) \equiv \int_0^\infty d\xi \frac{1}{\xi^{\gamma+2-(\beta/2)MN}} \exp\left(-\frac{1}{\xi}\right) \frac{\exp\left[-\xi\gamma\beta \mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]}{Z(\xi)}$$

$$\propto \left(\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right)^{\frac{1}{2}(\gamma+1-\beta NM/2)} K_{\gamma+1-\beta NM/2}\left(2\sqrt{\beta\gamma \mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)}\right)$$

$$P\ (\mathbf{X}) \longrightarrow P\ (\mathbf{X})$$

# Two distributions of matrix elements

$$P_1(\mathbf{X}) \propto \left(1 + \frac{\beta}{\gamma}\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right)^{-\gamma+\frac{1}{2}\beta NM}.$$

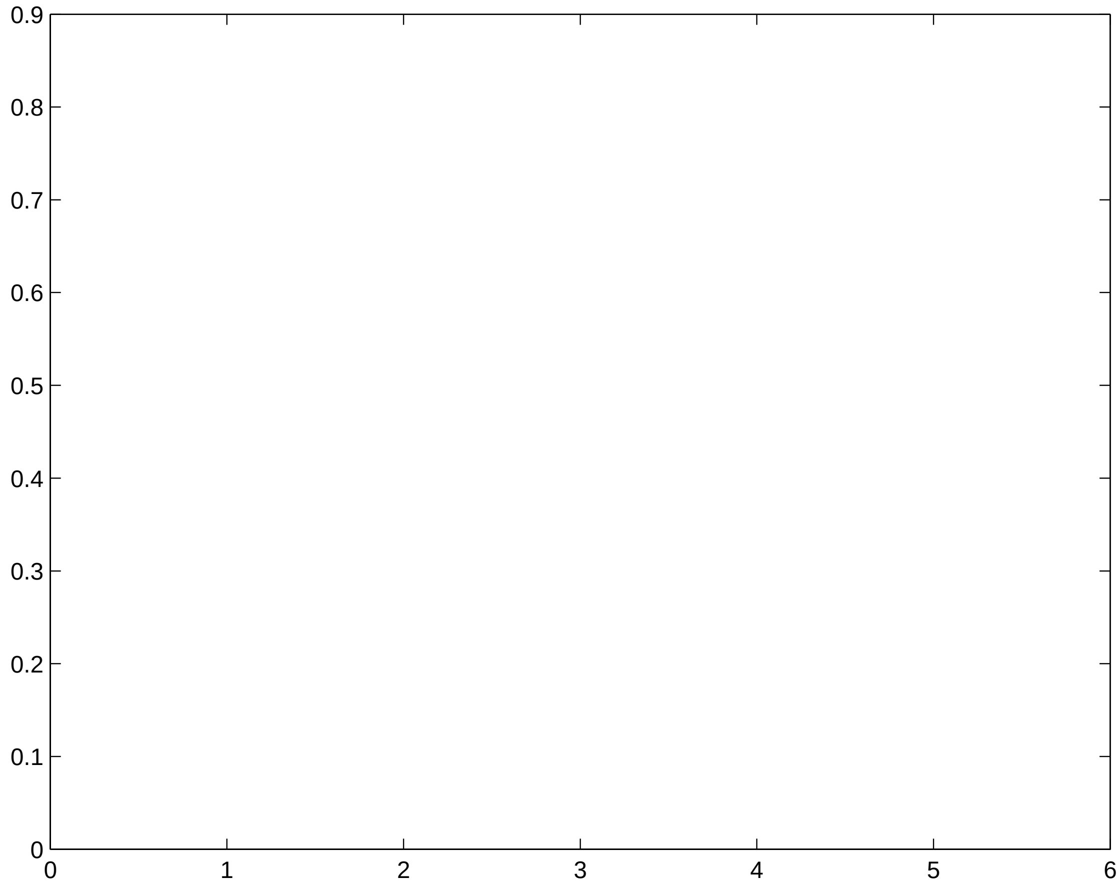
Power-law decay of spectral correlations

[illegible] $\left( \ldots \right)^{\frac{1}{2}(\gamma+1-\beta NM/2)}$ [illegible]

Exponential decay of spectral correlations

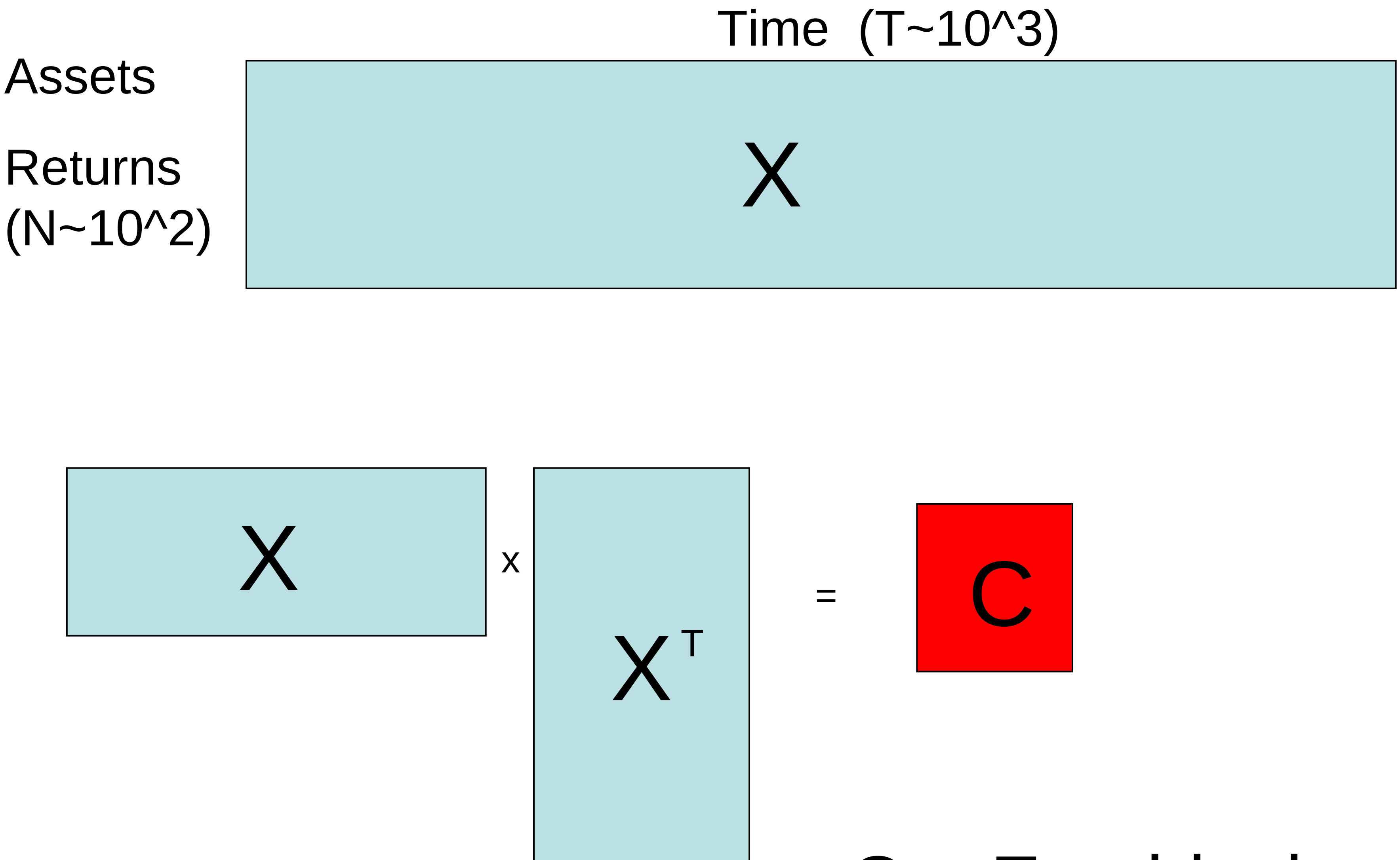
Both are obtained as averages of the standard Wishart-Laguerre weight over different distributions of the variance of **X**

$$\mathcal{P}_\gamma(\lambda_1, ..., \lambda_N) \propto \int_0^\infty d\xi \, \frac{1}{\xi^{\gamma+2-\frac{\beta}{2}NM}} \exp\left[-\frac{1}{\xi}\right] \mathcal{P}_{WL}(\lambda_1, ..., \lambda_N; \xi)$$


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5
6

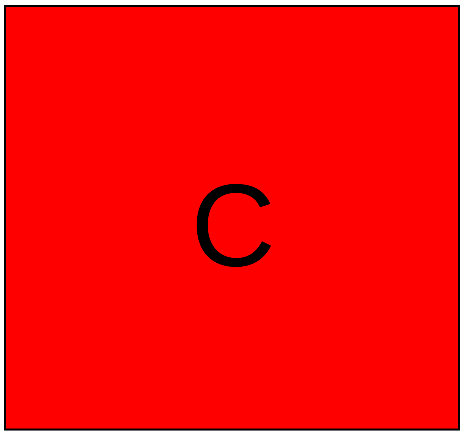
# Conclusions

- Random Covariance Matrices
- Variance of data fluctuates from one sample to another according to a normalized distribution $f(\eta)$
- Integral Transform of Wishart-Laguerre ensembles, depending on a single deformation parameter $\gamma$
- The model can be solved exactly
- Expected applications beyond the usual superstatistical classes




C = Empirical Covariance Matrix

# Empirical Covariance Matrix



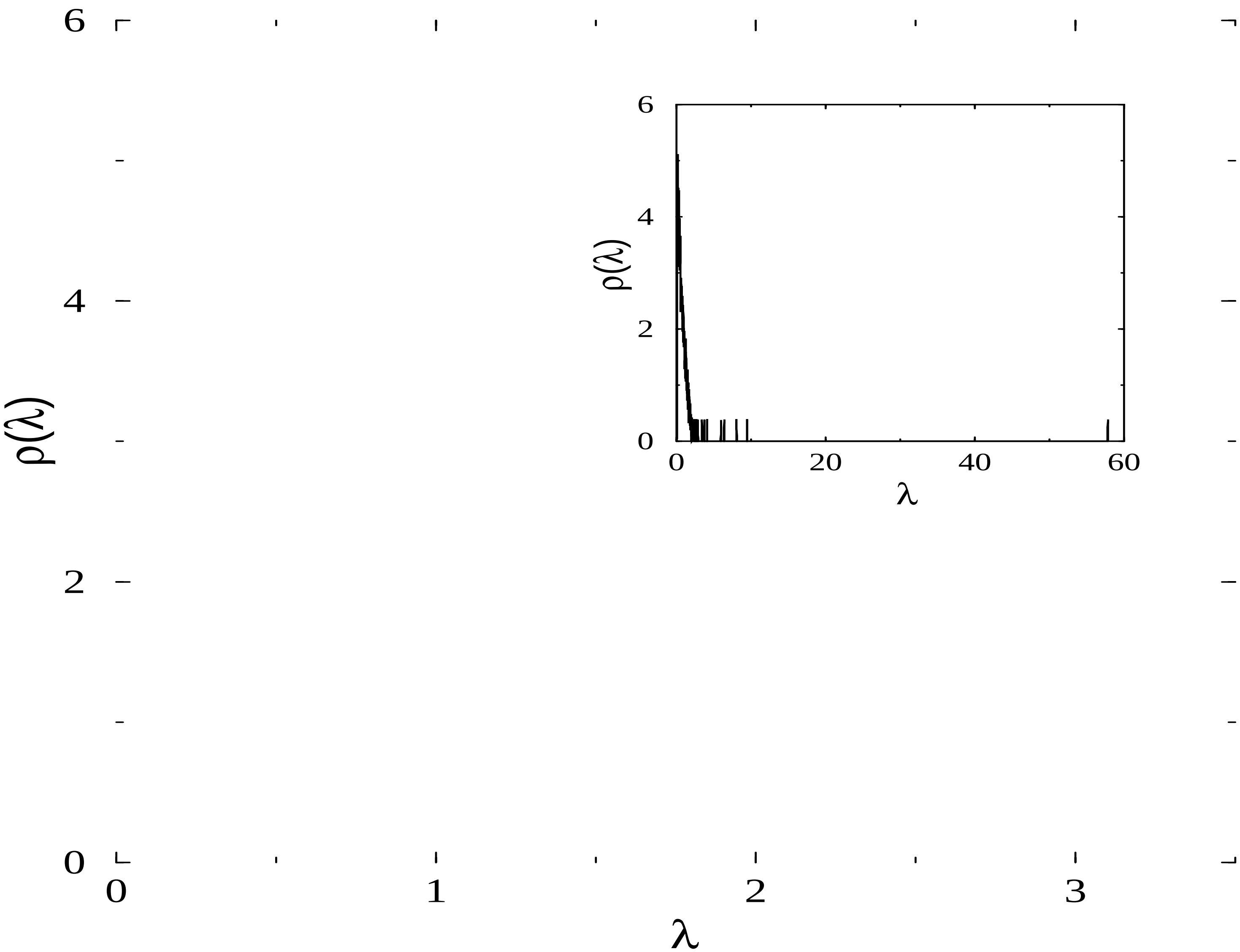

- N x N
- Real
- Symmetric
- Positive definite

Eigenvalues are real and positive

# What is the amount of randomness in financial data?

↓

Random Covariance Matrices: Wishart-Laguerre ensemble



6
4
2
0
0
1
2
3
ρ(λ)
λ
6
4
2
0
0
20
40
60
ρ(λ)
λ

# A new model of random covariance matrices

- Exactly solvable
- Recovers Wishart-Laguerre in a certain limit
- Power-law tails

$$P(\mathbf{X}^T\mathbf{X}) \quad \exp \quad \mathrm{Tr}\ \mathbf{X}^T\mathbf{X} \quad \det \quad \mathbf{X}^T\mathbf{X}$$

$$P\ (\mathbf{X}^T\mathbf{X}) \quad 1 \quad \frac{1}{\gamma}\mathrm{Tr}\ \mathbf{X}^T\mathbf{X} \quad \det \quad \mathbf{X}^T\mathbf{X}$$

# Salient features of the deformed model

- The data matrix **X** has entries correlated in an intricate way.
- It recovers Wishart-Laguerre in the limit of large **γ**.
- We can hope to get power-law tails.
- It remains to prove that it is exactly solvable!

# Gamma-integral Identity

$$(1 \quad \mathrm{Y}) \qquad \frac{1}{(\quad)} \, {}_{0} \, e \qquad {}^{1} e^{\quad \mathrm{Y}} \, \mathrm{d}$$

$$P \quad \mathbf{X}^{T}\mathbf{X} \qquad 1 \quad \frac{1}{\gamma} \quad \mathbf{X}^{T}\mathbf{X} \qquad \det \quad \mathbf{X}^{T}\mathbf{X}$$

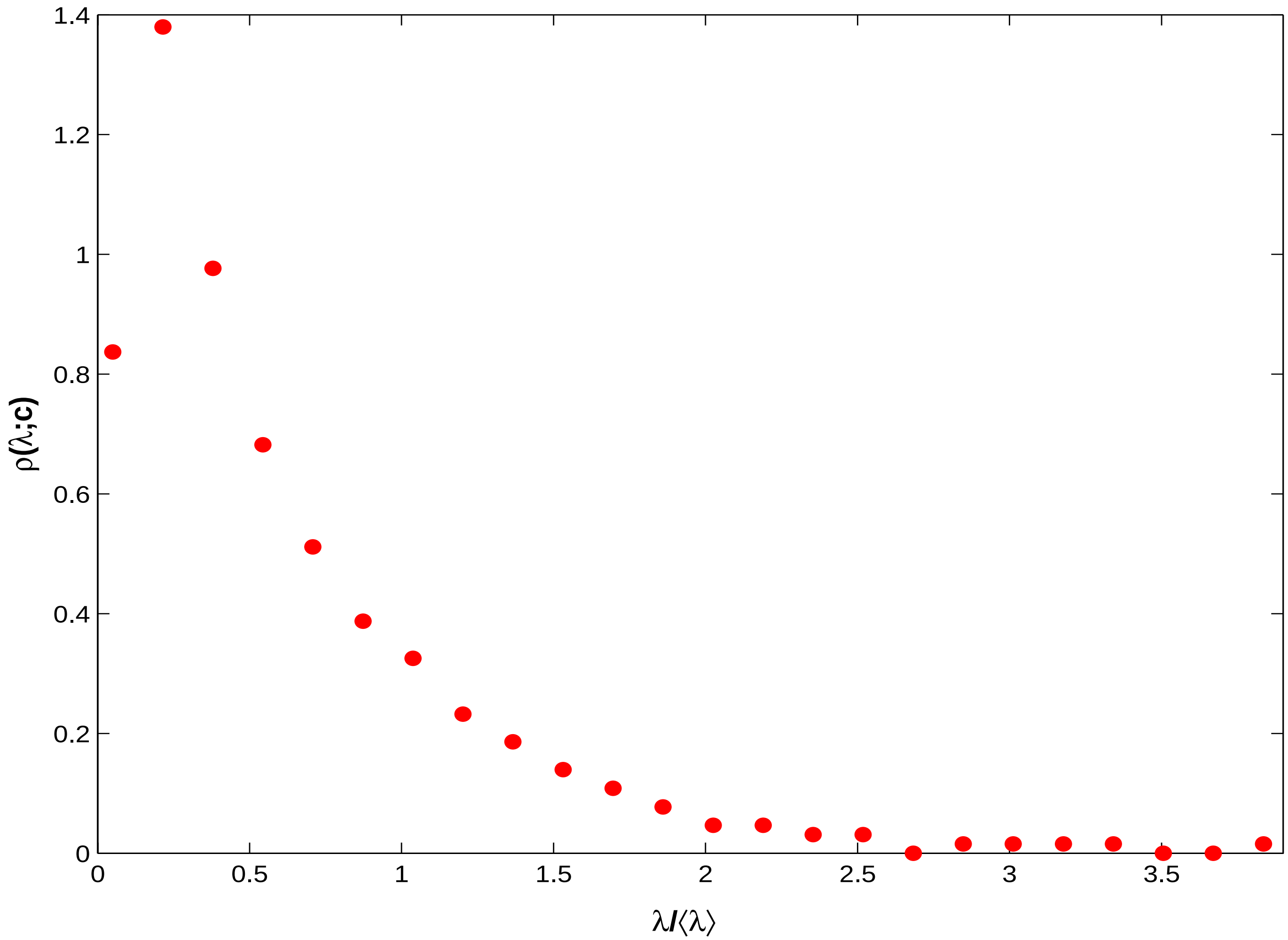
# Exact results

- Density of eigenvalues for finite N and **γ**

- Macroscopic density of eigenvalues in a certain double scaling limit

# Density of Eigenvalues

# Comparison to Financial Data



# Related works

- Z. Burda *et al.*, Phys. Rev. E **74**, 041129 (2006).
- A.C. Bertuola *et al.*, Phys. Rev. E **70**, 065102 (2004) .
- G. Biroli *et al.*, Acta Phys. Pol. **38**, 4009 (2007).

# Summary

- Exactly solvable deformation of Wishart-Laguerre ensemble of random matrices.
- Only one free parameter , such that we recover WL for
- Good agreement with eigenvalue distribution from financial data