The $O(n)$ model on random lattices of all topologies

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1 - $O(n)$ model

$$W_k^{(g)}(x_1, \dots, x_k) = \sum \frac{1}{\text{number of automorphisms}} \frac{t^v\, t_3^{n_3} \cdots t_d^{n_d}\, n^L\, z}{x_1^{j_1+1} \cdots x_k^{j_k+1}}$$

$\sum$ ranges over connected genus g discrete surface with k boundaries, built with:

- v vertices
- n_j j-gons ($j \geq 3 \geq d$, d fixed but arbitrary)
- triangles carrying a piece of path forming exactly L loops
- a marked j_i-gon ($j_i \geq 1$) with a marked edge as i-th boundary ($1 \leq i \leq k$)

For $n \in \mathbb{N}$, it admits a representation as a formal hermitian matrix model [1, 6]:

$$d\lambda(M, A_1, \dots, A_n) = \overbrace{dM\, dA_1 \cdots dA_n}^{\text{Lebesgue measure on } H_N^{1+n}} \exp\left(-\frac{N}{t}\mathrm{Tr}\left(V(M) + \sum_{i=1}^{n}\left(\frac{1}{2z} + M\right)A_i^2\right)\right)$$

$$\left\langle \prod_{x_i \in K} \mathrm{Tr}\frac{1}{x_i - M} \right\rangle_c = \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-k} W_k^{(g)}(K)$$

- $V(x) = \frac{x^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} x^j$
- equalities between formal series in t, of polynomials in t_j's and rational functions in x_i's
- c for cumulant
- analytic continuation for $n \in \mathbb{R}$

We proved an algorithm to compute all $W_k^{(g)}$'s

2 - Interest for the $O(n)$ model

Critical points

Exhibits critical points [2] different from pure gravity at $t_c > 0$, for $0 < |n| \leq 2$

For $n = -2\cos \pi g$ ($g \in\,]0, 1[$), several continuum limits

Believed to be $\mathrm{CFT}_c \otimes$ gravity, with

$c = 1 - 6\left(\tilde{g}^{1/2} - \tilde{g}^{-1/2}\right)^2$ where $\tilde{g} = \varepsilon(1 - g) + 2p + 1$

→ reach non rational CFT's by the continuum limit of a microscopic model

Combinatorics → Counting discrete surfaces with additional structure

Duality to $q = n^2$ Potts model

Fully packed loops ↔ dimer configurations when $V(x) = x^2/2$

Matrix models → A direction of generalization of the algebraic geometry tools developed for the 1-matrix model ($n = 0$) [5]

3 - The method of loop equations

Loop equations = change of variables in the matrix integrals

Powerful way to prove automatically combinatorial recursion relations [7]

Lemma 0

When $x \to \infty$ $\begin{cases} W_1^{(0)}(x) & t/x \\ W_k^{(g)}(x, J) & O(1/x^2) \text{ else} \end{cases}$

Combinatorial lemma

in each variable (for $k \geq 1$), $W_k^{(g)}(x_1, \dots, x_k)$ is holomorphic with one cut $[a(t), b(t)] \subset \mathbb{C}$

When $x \to a_i \in \{a(t), b(t)\}$ $W_1^{(0)}(x) - W_1^{(0)}(a_i) \sim \sqrt{(x - a_i)}$

There exists a set of loop equations determining uniquely $W_k^{(g)}$ satisfying these analytical properties

4 - Analytical properties of $W_k^{(g)}$

The one-cut property implies $x \in [a(t), b(t)]$ and $\to 0$:

[3] $W_1^{(0)}(x + i\,) + W_1^{(0)}(x - i\,) + n W_1^{(0)}(-x) = V(x)$

[3] $W_2^{(0)}(x_1 + i\,, x_2) + W_2^{(0)}(x_1 - i\,, x_2) + n W_2^{(0)}(-x_1,$

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