

Critical asymptotics for Toeplitz determinants

Tom Claeys

Université de Lille 1

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Joint work with A. Its and I. Krasovsky

Toeplitz determinants

- Toeplitz matrix = matrix which is constant along diagonals

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{-2} \\ \vdots & \ddots & \ddots & c_0 & c_{-1} \\ c_{n-1} & \dots & c_{-2} & c_{-1} & c_0 \end{pmatrix}$$

- Toeplitz determinant is the determinant of a Toeplitz matrix
- Asymptotics for Toeplitz determinants when the size of the matrices tends to infinity?

Toeplitz determinants



Toeplitz determinants

- If the weight f
 - ▶ is "smooth"
 - ▶ has no zeros
 - ▶ has a continuous logarithm (winding number around the origin)
- Szegő's strong limit theorem: as $n \rightarrow \infty$,

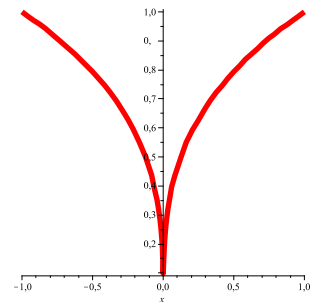
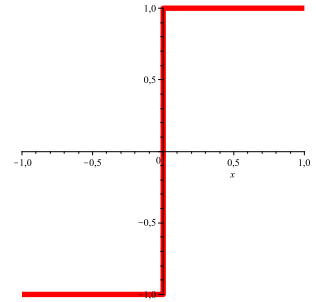
$$\ln D_n f = n \ln f_0 + \sum_{k=1}^n k \ln f_k \ln f_{-k} + o(1),$$

with

$$\ln f_k = \frac{1}{2\pi} \int_0^{2\pi} \ln f(e^{i\theta}) e^{-ik\theta} d\theta.$$

Fisher-Hartwig singularities

- Two types of weights for which Szegő asymptotics are not valid
 - ▶ jump discontinuities
 - ▶ root type singularities



■ Example

$$f(e^i) = \cos(e^i) e^i \left(\frac{1}{2} \right) e^{V(e^i)}, \quad \text{for } \alpha < \beta < \pi,$$

$$\text{with } \operatorname{Re} \alpha > 1$$

Fisher-Hartwig singularities

- For weights with one Fisher-Hartwig singularity with parameters α (root) and β (jump),

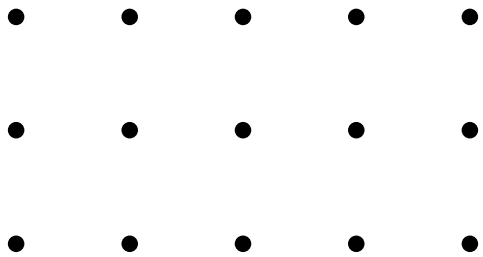
$$\ln D_n f = nV_0 + \sum_{k=1}^{\infty} kV_k V_{-k} + \sum_{k=1}^{\infty} V_k + \sum_{k=1}^{\infty} V_{-k} + \frac{2}{n} \ln n + \ln \frac{G(\alpha + \beta) G(\alpha + \beta + 1) \dots G(\alpha + \beta + n - 1)}{G(\alpha) G(\alpha + 1) \dots G(\alpha + n - 1)} + o(1),$$

as $n \rightarrow \infty$, where G is Barnes' G -function, and

$$V_k = \frac{1}{k} \left(\alpha + \beta - \frac{1}{2} \right)$$

2d Ising model

- lattice with an associated spin variable taking values ± 1 at each point of the lattice



2d Ising model

- 2-spin correlation functions are Toeplitz determinants:

$$\langle \sigma_{00} \sigma_{0k} \rangle = D_k f ,$$

for a certain symbol f

- ▶ For $T < T_c$

Transition from Szegő to FH

Transition from Szegő to FH

- Asymptotics as n

Transition from Szegő to FH

$$\int_0^{2nt} \left(w(x) - \frac{v(x)^2}{x} \right) dx + \int_{+\infty}^{2nt} \left(w(x) - \frac{v(x)^2}{x} \right) dx + \ln nt,$$

■ v

Transition from Szegő to FH

Asymptotics

$$v(x) \begin{cases} \mathcal{O} \left(\frac{1}{x^2} + \mathcal{O} \left(\frac{1}{x^2} \ln x \right) \right), & x \rightarrow \infty, \\ \mathcal{O} \left(e^{-cx} \right), & x \rightarrow +\infty, \end{cases}$$

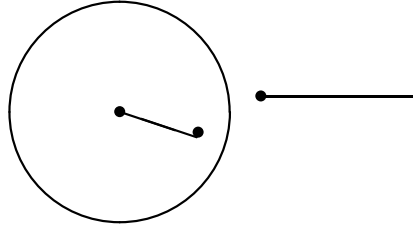
$$\int_0^+ v(x) dx = \frac{1}{2} + \frac{1}{2}.$$

$$w(x) \begin{cases} \frac{1}{x} + \mathcal{O} \left(\frac{1}{x^2} + \mathcal{O} \left(\frac{1}{x^2} \ln x \right) \right), & x \rightarrow \infty, \\ \mathcal{O} \left(e^{-cx} \right), & x \rightarrow +\infty. \end{cases}$$

$$x \begin{cases} \frac{1}{2} + \frac{1}{2} \ln x + \mathcal{O} \left(\frac{1}{x} \right), & x \rightarrow \infty, \\ \ln \frac{\Gamma(1 + \frac{1}{2}) \Gamma(1 - \frac{1}{2})}{\Gamma(1 + 2 \cdot \frac{1}{2})} + \mathcal{O} \left(e^{-cx} \right), & x \rightarrow +\infty. \end{cases}$$

Transition from Szegő to FH

- Extension to complex t ?



Expansion is valid for $\arg t < \frac{\pi}{2}$ if contour of integration does not contain poles of w

- different choices of contour different branches of logarithm

Transition from Szegő to FH

- what if $\text{Im } w(x)$ and/or $\text{Re } w(x) < 0$?
 - ▶ $w(x)$ is not real for $x > 0$
 - ▶ w can have poles on \mathbb{R}_+
 - ▶ asymptotic expansion holds only if we integrate over a pole-free contour
 - expansion not valid if nt is a pole of $w(x)$
 - ▶ poles correspond to Toeplitz determinants approaching 0
 - different choices of integration contour
 - expansion picks up residue of w
 - residue of w



Orthogonal polynomials

- Heine's formula: determinant formula for orthogonal polynomials

$\rho_{n\alpha}$



Asymptotics for Toeplitz determinants

General approach to obtain asymptotics for Toeplitz determinants for weight f

- Step 1: deform weight f smoothly to a weight for which Toeplitz determinant is known (e.g. uniform weight),

$$f_t z, \quad f_1 z, \quad f, \quad f_0 z$$

- Step 2: try to find **differential identity** for $\frac{d}{dt}$

Transition from Szegő to FH

Applied to our transition between Szegő and FH

- Step 1: deformation of weight:

$$f_t(z) = z e^t + z e^{-t}$$

Transition from Szegő to FH

■ Step 2: differential identity

$$\frac{d}{dt} \ln D_n(t) + e^t (Y^{-1} Y')_{22} e^t + e^{-t} (Y^{-1} Y')_{22} e^{-t}$$

where

$$Y(z) = \begin{pmatrix} p_n^{-1}(z) & C_1 \frac{p_n(z) f(z) dz}{-z} \\ z^{n-1} p_{n-1}(z) & C_1 \frac{p_{n-1}(z) f(z) dz}{-z} \end{pmatrix}$$

- ▶ Y is solution of the Riemann-Hilbert problem for orthogonal polynomials

