The Chiral Gaussian Two-Matrix Ensemble of Real Asymmetric Matrices

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Definition of the New Ensemble

- Solve for eigenvalue correlation functions of the matrix

$$D \equiv \begin{pmatrix} 0 & P + \mu Q \\ P^T - \mu Q^T & 0 \end{pmatrix}$$

- P and Q are real-valued matrices (i.e. $\beta = 1$) of size $N \times (N + \nu)$, with elements independently-distributed Gaussian $\sim N[0, 1]$.
- $\mu \in [0, 1]$ is the non-Hermiticity parameter. $\mu = 0$, chGOE [3].
- Model has ν eigenvalues which are precisely zero.
- Other eigenvalues come in
  - pairs $(\pm\Lambda_j)$, either both real or both imaginary, or
  - complex-valued quadruplets $(\pm\Lambda_j, \pm\Lambda_j^*)$.
- Model finds applications in 2-colour Quantum Chromodynamics ([illegible]

The Joint PDF (P_N) for Finite N

We solve for the eigenvalues of the $N \times N$ **Wishart-type** matrix:

$$W \equiv (P + \mu Q)(P^T - \mu Q^T)$$

whose entries are **non-Gaussian** and **correlated**. The Dirac eigenvalues are the square roots of Wishart eigenvalues ($z = x + iy = \Lambda^2$).

$N = 1$ One real (Wishart) eigenvalue

$$P_1(x) \sim |x|$$

The Algebraic Structure of ¯ = 1 Ensembles

P_N has the same form as the non-chiral Ginibre crossover ensemble (see [11, 12] and references therein). Therefore, we can immediately write the partition function Z and correlation functions R_n as follows:

$$Z \;» \;\mathrm{Pf}\; {}^{\cdot Z}\; d^2z_1 \;{}^{Z}\; d^2z_2\, F(z_1;z_2)\, z_1^{i\,i\,1} z$$

Calculation of the Kernel using Grassmann Integrations

We can show that the kernel K_N is related to the expectation of the product of two characteristic polynomials:

$$K_N(u,v) \sim (u - v)\frac{\langle \det(\sqrt{u} - D)\det(\sqrt{v} - D)\rangle_{N-2}}{(uv)^{o/2}}.$$

To evaluate this expectation:

- Replace the determinants with Grassmann integrals

$$\det M = \int d\eta\, e^{-\eta_i^* M_{ij}\eta_j}.$$

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Eigenvalue Densities at Finite N

The Strong Large-N Limit

For physical applications, large-N limits are usually required. For the strong limit, we keep μ fixed. No scaling of eigenvalues is required if we are interested in behaviour close to the origin. Using the Hardy-Hille formula (for weighted sums of Laguerre polynomials), we find:

$$\rho^C(z) \sim -2i\,\mathrm{sgn}(\mathrm{Im}\, z)\, 2 \int_0^\infty \frac{dt}{t} e^{-\eta_+^2 t (z^2 + z^{*2}) - \frac{1}{4t}} K_{\nu/2}(2\eta_+^2 t |z|^2)\, \mathrm{erfc}(2\eta_+ \sqrt{t} |\mathrm{Im}\, z|)$$
$$\Theta(z - z^*)$$

The Weak Large-N Limit

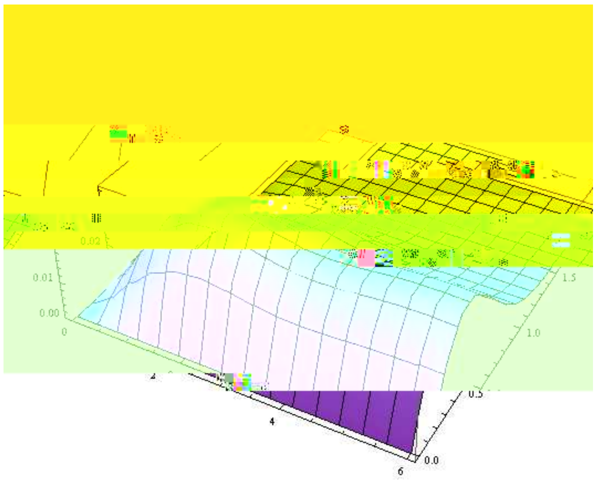
Here, we scale μ with $\frac{1}{\sqrt{N}}$ (i.e. $\mu \to 0$). It is also necessary to scale the Dirac eigenvalues by $\frac{1}{\sqrt{N}}$ if we are to reach an interesting limit:

$$\mu \equiv \frac{\alpha}{\sqrt{2N}} \quad \text{for fixed } \alpha;$$
$$z \equiv \frac{\hat{z}}{4N} \quad \text{for fixed } \hat{z}.$$

We can determine the complex density by combining the limiting weight function with the limit of the kernel:

$$K^W(\hat{z}_1; \hat{z}_2) \propto \int_0^1 ds\, s^2 e^{-2\alpha^2 s^2} \left\{ \sqrt{\hat{z}_1} J_{\nu+1}(s\sqrt{\hat{z}_1}) J_\nu(s\sqrt{\hat{z}_2}) - (\hat{z}_1 \leftrightarrow \hat{z}_2) \right\}.$$

We show here $\alpha^2 = 0.2$ (with $\nu = 0$):



- We see that the complex eigenvalues almost all occupy a strip of finite width, parallel to the real axis.

But for the real and imaginary densities, we find (schematically)

$$\rho_\infty \equiv \lim_{N\to\infty} \rho_N = \lim_{N\to\infty} \int K_N \times F \neq \int \lim_{N\to\infty} K_N \times F.$$

We will return to this issue in the (F)-377(6)]T0TuF((F)34Tf7.170TD[(:)]TJE

Review