Random matrices with equi-spaced external source

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joint work with Dong Wang (University of Singapore)

Random matrices with external source

- space of **n × n** Hermitian matrices with probability measure

$$\frac{}{\mathbf{Z}_n} \operatorname{xp}\left(-\mathbf{n}\mathrm{Tr}\left(\mathbf{V}\left(\mathbf{M}\right.\right.\right. \quad - \mathbf{AM} \quad \mathbf{dM},$$

where

- **V** is a polynomial of even degree with positive leading

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- if **A** , unitary ensemble

$$\frac{}{\mathbf{Z}_n} \text{xp}(-\mathbf{n}\,\mathrm{Tr}\,\mathbf{V}(\mathbf{M} \quad \mathbf{dM}.$$

- we will study the case

$$\mathbf{A} \quad \frac{}{\mathbf{n}}\mathrm{diag}(\ , \ , \ldots, \mathbf{n} -$$

  - for $\mathbf{V}(\mathbf{x}$ $\mathbf{cx}^2$, eigenvalues behave like **n** non-intersecting Brownian motions starting at and ending at $, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$

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- Joint probability distribution of eigenvalues in the ensemble

$$\frac{}{\mathbf{Z}_n} \ \mathrm{xp}\big(-\mathbf{n}\,\mathrm{Tr}\,\big(\mathbf{V}\,\big(\mathbf{M} \quad - \mathbf{AM} \quad \mathbf{dM}$$

is given by

$$\frac{}{\mathbf{Z}_n}\Big| \ \mathrm{t}\big(\mathbf{e}^{na_{\mathbf{i}}\lambda_{\mathbf{j}}} \ {}_{i,j=1,\dots,n}\Big| \ \prod_{i<j} |\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j| \ \prod_{j=1}^{n} \mathbf{e}^{-nV(\lambda_{\mathbf{j}})}\mathbf{d}\boldsymbol{\lambda}_j$$

  - if $\mathbf{A} \quad \frac{1}{n}\mathrm{diag}\big(\ , \ , \dots , \mathbf{n} - \quad ,$

$$\frac{}{\mathbf{Z}_n} \prod_{i<j} \big(\boldsymbol{\lambda}_i - \boldsymbol{\lambda}_j \ \prod_{i<j} \big(\mathbf{e}^{\lambda_{\mathbf{i}}} - \mathbf{e}^{\lambda_{\mathbf{j}}} \ \prod_{j=1}^{n} \mathbf{e}^{-nV(\lambda_{\mathbf{j}})}\mathbf{d}\boldsymbol{\lambda}_j.$$

- Behavior of eigenvalues for large **n**?

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- $\mathbf{A} \quad \mathrm{diag}(\mathbf{a}, \dots, \mathbf{a}, -\mathbf{a}, \dots, -\mathbf{a}$ *(Bleher-Kuijlaars, Bleher-Delvaux-Kuijlaars, Adler-van Moerbeke)*
  - vector equilibrium problem
  - critical point: Pearcey kernel
- $\mathbf{A} \quad \mathrm{diag}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}\ ,\ \ , \dots,$ with **k** fixed *(Baik-Wang, Bertola-Buckingham-Lee-Pierce, Adler-Délépine-van Moerbeke)*
  - every non-zero eigenvalue of **A** is responsable for at most one outlier-eigenvalue of **M**
- External source matrix with **n** different eigenvalues *(Eynard-Orantin)*

External source

- **A** $\frac{1}{n}$diag(, ,...,n− ,n−

Limiting mean eigenvalue density

- limiting mean distribution minimizes

$$-\iint \log|t-s|^{-1}d\mu(t)\,d\mu(s) \quad - \iint \log|e$$

Eigenvalue correlation kernel

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A $\frac{1}{n}$diag(, , ..., n − , n − :

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y} \quad \mathbf{e}^{-\frac{n}{2}V(x)}\mathbf{e}^{-\frac{n}{2}V(\;)} \sum_{=0}^{n-1} \mathbf{p}\;(\mathbf{x}\;\mathbf{q}\;(\mathbf{e}$$

  - polynomials $\mathbf{p}_k$ of degree **k** and $\mathbf{q}_j$ of degree **j** are determined by the orthogonality conditions

$$\int \mathbf{p}_k(\mathbf{x}\;\mathbf{q}_j(\mathbf{e}^x\;\;\mathbf{e}^{-nV(x)}\mathbf{dx} \quad \boldsymbol{\delta}_{kj}$$

  - $\mathbf{p}_k$'s are type II multiple OPs with **n** orthogonality weights
    $, \mathbf{e}^x, \mathbf{e}^{2x}, \ldots, \mathbf{e}^{(n-1)x}$

Eigenvalue correlation kernel

Random matrices with external source

$\mathbf{A} \quad \frac{1}{n}\text{diag}(\; , \; , \ldots , \mathbf{n} - \; , \mathbf{n} - \quad :$

- correlation kernel for eigenvalues is given by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y} \qquad \mathbf{e}^{-\frac{n}{2}V(x)}\mathbf{e}^{-\frac{n}{2}V(\;)} \sum_{=0}^{n-1} \mathbf{p} \; (\mathbf{x} \; \mathbf{q} \; (\mathbf{e}$$

  - polynomials $\mathbf{p}_k$ of degree $\mathbf{k}$ and $\mathbf{q}_j$ of degree $\mathbf{j}$ are determined by the orthogonality conditions

$$\int \mathbf{p}_k(\mathbf{x} \; \mathbf{q}_j(\mathbf{e}^x \;\; \mathbf{e}^{-nV(x)}\mathbf{dx} \qquad \boldsymbol{\delta}_{kj}$$

  - $\mathbf{q}_j$'s are related to type I multiple orthogonal polynomials

Eigenvalue correlation kernel

Interpretation of the polynomials in terms of the random matrix ensemble

$$\frac{1}{Z_n} \exp(-n \operatorname{Tr}(V(M) - AM))\, dM$$

or the determinantal point process

$$\frac{1}{Z_n} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j .$$

- p

Eigenvalue correlation kernel

Eigenvalue correlation kernel

- RH problem for usual OPs *(Fokas-Its-Kitaev '92)*

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,

(c) $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$,

- Unique solution given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{p_n(s) w(s)}{s - z} ds \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(s) w(s)}{s - z} ds \end{pmatrix},$$

Eigenvalue correlation kernel

- polynomials defined by

$$\int_{\mathbb{R}} p(x)\, q(e^x)\, e^{-nV(x)}\,dx \quad \delta$$

- standard RH problem for MOPs is of size **n** - inconvenient for **n** large
- let

$$Y_1(z) \quad \kappa_n^{-1} p_n(z)$$

and

$$Y_2(z) \quad \frac{-\kappa_n^{-1}}{\pi i} \int_{\mathbb{R}} \frac{p_n(s)}{e - e^s} e^{-nV(s)}\,ds.$$

-

RH problem for polynomials

1. $\mathbf{Y}$ $(\mathbf{Y}_1, \mathbf{Y}_2$, where $\mathbf{Y}_1$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and $\mathbf{Y}_2$ is analytic in

RH problem for polynomials

- there is also a × matrix RH problem
  - unlike for usual orthogonal polynomials, $\mathrm{t}\,\mathbf{Y}(\mathbf{z}$ □
  - taking inverses is not possible
  - no advantage
- there is a dual RH problem for $\mathbf{Y}$ $(\mathbf{Y}_1, \mathbf{Y}_2$, where

$$\mathbf{Y}_1 \quad -\kappa_n q_n(e \quad , \qquad \mathbf{Y}_2(z \quad \frac{-\kappa_n}{\pi i} \int \frac{q_n(e^s}{z - s} e^{-nV(s)} ds.$$

RH problem for polynomials

1. $\mathbf{Y}$ $(\mathbf{Y}_1, \mathbf{Y}_2$, where $\mathbf{Y}_2$

RH problem for polynomials

- Asymptotic analysis of the RH problem if the support of **μ** is one interval: Deift/Zhou steepest descent analysis
- Modifications compared to analysis for OPs
  - construction of two **g**-functions

$$g(z) = \int \log(z - y)\, d\mu(y)$$

$$g(z) = \int \log(e^{z} - e^{y})\, d\mu(y).$$

  - Crucial step: transformation of the RH problem to a non-local scalar RH problem in the complex plane

RH problem for polynomials

- Transformation to shifted RH problem of the form

1. $\mathbf{F}\ \ \mathbb{C}\setminus \quad \to \mathbb{C}$ is analytic
2. for $\mathbf{z}$ □ , we have

$$\mathbf{F}_+(\mathbf{z} \quad \mathbf{F}_-(\mathbf{z}\ \mathbf{J}_n(\mathbf{z} \quad \mathbf{F}_\pm(\mathbf{f}(\mathbf{z}\ \ \mathbf{J}_n(\mathbf{z}\ ,$$

   with $\mathbf{f} \quad \to \quad ,$

3.

Outlook

- Universality
  - sine kernel
  - Airy kernel
- multi-cut case
- large **n** behavior in more general point processes of the form

$$\frac{1}{Z_n}\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i<j}(f(\lambda_i)-f(\lambda_j))\prod_{j=1}^{n}e^{-nV(\lambda_j)}d\lambda_j.$$