

From Classical to Quantum Integrable Systems

Panagiota Adamopoulou and [Clare Dunning](#)
University of Kent

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The Ordinary Differential Equation / Integrable Model Correspondence

The
eigenvalues
of certain ordinary differential equations
are the Bethe roots of certain massless quantum integrable models

IM
ODE

Main result

The
eigenvalues
of linear systems of certain partial differential equations
are the **Bethe roots** of certain massive quantum integrable models
IM
ODE

Lukyanov, Zamolodchikov; Dorey, Faldella, Negro, Tateo;
Adamopoulou, TCD; Ito, Locke

Ordinary Differential Equations

The eigenvalues $f(E_j)$ of

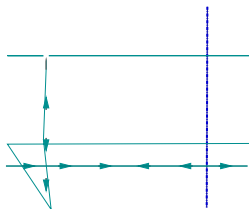
$$\frac{d^2}{dx^2} + (x^{2M} - E) + \frac{l(l+1)}{x^2} \quad (x; E; l) = 0 \quad \int L^2(\mathbb{R}^+)$$

satisfy **Bethe ansatz equations**

$$\prod_{k=1}^J \frac{E_k - l^2 E_j}{E_k + l^2 E_j} = e^{\frac{i(2l+1)}{M+1}} \quad j = 1; 2; \dots; J$$

where $l = e^{\frac{i}{M+1}}$

6-vertex model



The **same** Bethe ansatz equations encode the ground state of the 6-vertex model with twisted boundary conditions in the thermodynamic limit

Spectral parameter E

$$\text{Anisotropy} = \frac{M}{2M+1}$$

$$\text{Twist} = \frac{(2l+1)}{2M+2}$$

XXZ model

The **same** Bethe ansatz equations encode the ground state of the XXZ model

$$H_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^N \left(x_j^2 + x_{j+1}^2 + y_j^2 + y_{j+1}^2 + z_j^2 + z_{j+1}^2 \right) \cos 2\theta$$

with twisted boundary conditions

$$z_{N+1} = z_1 ; \quad x_{N+1} = e^{i\theta} x_1 ; \quad y_{N+1} = e^{-i\theta} y_1$$

in the thermodynamic limit

Massless quantum field theory

The **same** Bethe ansatz equations encode the primary fields of a $c = 1$ conformal field theory or equivalently the massless limit of the quantum sine-Gordon model

Central charge $c(M) = 1 + \frac{6M^2}{M+1}$

Highest weight $(M; l) = \frac{(2l+1)^2 - 4M^2}{16(M+1)}$

Vacuum parameter $\rho = \frac{2l+1}{4M+4}$

Massive sine-Gordon model

Massive model is defined on a cylinder with radius s

A specific solution of the modified sinh-Gordon equation

Require a unique, real solution of

$$\partial_z \bar{\partial}_{\bar{z}} \phi(z; \bar{z}) - e^{2\phi(z; \bar{z})} + p(z)\bar{p}(\bar{z}) e^{-2\phi(z; \bar{z})} = 0$$

that is finite everywhere except at $z = 0$ where

$z = e^i; \bar{z} = e^{-i}$ with

- | Periodicity

$$\phi(z; \bar{z}) + \phi(e^{i+2\pi} z; e^{-i-2\pi} \bar{z}) = \phi(z; \bar{z}) + M$$

- | Asymptotic behaviour as $|z| \rightarrow 0$:

$$\phi(z; \bar{z}) = M \ln |z| + \phi_0 + O(|z|)$$

- | Asymptotic behaviour as $|z| \rightarrow \infty$

$$\phi(z; \bar{z}) = M - 2 \ln |z| + o(1)$$

The associated linear problem

is

$$@_z + U(z; z) = 0; \quad @_z + V(z; z) = 0$$

where $\begin{pmatrix} 1 \\ 2 \end{pmatrix}^T$

The solution to the linear system is

$$(z; z) = \begin{pmatrix} e^{-2} \\ 1 - 2e^{-2} \end{pmatrix} = e^{-2} \begin{pmatrix} 1 \\ 1 - 2 \end{pmatrix} = e^{-2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eliminating z_2 we find

$$\left[\begin{array}{cc} @_z^2 + (@_z)^2 & @_z^2 + 2p(z) \end{array} \right] (z; z) = 0$$

Similarly

$$\left[\begin{array}{cc} @_z^2 + (@_z)^2 & @_z^2 + 2p(z) \end{array} \right] (z; z) = 0$$

Classical integrability

The modified affine Toda equations arise from the zero-curvature condition $V_z - U_z + [U; V] = 0$ of the linear problem

$$\partial_z + U(z; z; \lambda) = 0; \quad \partial_z + V(z; z; \lambda) = 0$$

with Lax matrices

$$(U(z; z; \lambda))_{ij} = \partial_z \delta_{ij} + (C(z))_{ij}$$

and

$$(V(z; z; \lambda))_{ij} = \partial_z \delta_{ij} + \lambda^{-1} (C(z))_{ji}$$

$$(C(z))_{ij} = \begin{cases} \exp(\lambda^{-1} \sum_{j=1}^i \alpha_j) & ; j = 1; \dots; n-1 \\ p(z) \exp(\lambda^{-1} \sum_{j=1}^i \alpha_j) & ; j = n; \end{cases}$$

where

$$C_{ij} = \begin{cases} 1 & \text{if } i \equiv j \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Associated linear problem

The linear problem

$$\partial_z + U(z; z;) = 0; \quad \partial_{\bar{z}} + V(z; z;) = 0$$

has solutions

$$\begin{aligned} \psi_i(z; z;) &= e^{i \int z} \partial_z^{i+1} \psi + \partial_z^{i+1} \psi \\ &= e^{-i \int z} \partial_z^{i+1} \psi \end{aligned}$$

Eliminating $\psi_1, \dots, \psi_{n-1}$ or ψ_2, \dots, ψ_n ; we obtain

$$(\partial_z)^{n+1} \psi + p(z) \psi = 0;$$

$$(\partial_z)^{n+1} \psi + \bar{p}(z) \psi = 0$$

with n^{th} -order differential operators

$$D_n(\psi) = (\partial_z + 2\partial_z^{-1})(\partial_z + 2\partial_z^{-2}) \dots (\partial_z + 2\partial_z^{-n});$$

$$D_n(\psi) = (\partial_z - 2\partial_z^{-n}) \dots (\partial_z - 2\partial_z^{-2})(\partial_z - 2\partial_z^{-1}).$$

Massive to massless

In the *massless* limit

$$z \neq 0 ; z \rightarrow s \neq 0 ; \ln \rightarrow 1$$

with

$$x = z e^{\overline{M+1}} ; x = z e^{-\overline{M+1}} ; E = s^{nM} e^{\frac{n}{M+1}}$$

yields

$$\left((-1)^{n+1} D_n(\mathbf{g}) + p(x; E) \right) (x; E) = 0 ; p(x; E) = x^{nM} E$$

where

$$D_n(\mathbf{g}) = \partial_z \frac{g_{n-1}}{x} \frac{(n-1)}{x} \partial_z \frac{g_{n-2}}{x} \frac{(n-2)}{x} \partial_z \frac{g_0}{x}$$

This is precisely the n^{th} -order ODE appearing in the **massless** A_{n-1} ODE/IM correspondence

