How many stable equilibria will a large complex system have?

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after joint work with Yan Fyodorov PNAS 2016 and

Motivation

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Context: diversity vs. stability debate in theoretical ecology, in 1950-60s. Quantitative analysis wanted.

May's neighbourhood stability analysis Consider $\dot{} = (\ )\ \in\ ^{N}$ near equilibrium at $=$ a model for generic eco systems with degrees of freedom. By Taylor-expanding $(\ )$ in the neighbourhood of equilibrium

$\infty\propto\infty\in\in\forall\forall\in\dot{\infty}=\infty\infty\nabla\propto\in\forall\infty\infty\infty\infty\in\infty$

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May-Wigner instability transition

Have Girko's circular law: As $\to \infty$ EV distribution of $\sqrt{}$ converges to unif distrib on the unit disk Girko1984 Bai1997 Götze & Tikhomirov Tau & Vu 2010.

Also the rescaled spectral radius of is $\leq$ 1 in the limit of large Geman 1984.

Hence for large

the linearised system is stable if $\frac{\mu}{\alpha\sqrt{N}}$ **1** and **unstable if** $\frac{\mu}{\alpha\sqrt{N}}$ **1**.

In May's words: "The central feature of the above results for large systems is the very sharp transition from stable to unstable behaviour as the complexity ... exceeds a critical value". This statement is known as the May-Wigner theorem.

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In other words the linear framework despite being so popular gives no answer to the question about what is happening to the **original** system when it loses stability. Instability does not imply lack of persistence ... Populations operating out of equlibrium ... Limit cycles ...

Is there a signature of May-Wigner instability transition on the global scale?

Non-linear systems: setup

A simple model for generic large complex systems: consider

$$\frac{\ \ _i}{\ } = - \ \ _i + \ _i(\ _1, \ldots, \ _N) \qquad = 1, \ldots, \ .$$

0 as before and now ( ) is a smooth random field with components.

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This system may have multiple equilibria depending on the realisation of $(\ )$.

Near equilibrium $\quad_e$ it reduces to May's model with $= - \quad_e \quad_{jk} = \frac{f_j}{x_k}(\quad_e)$.

Gradient-descent flow, $= -\nabla$ **, is special (but typical) case.** Have

$$\frac{\quad}{\quad} = -\nabla \qquad (\ ) = \frac{|\ |^2}{2} + (\ ) \qquad \text{[note that } \quad_{jk} = \quad_{kj} \text{ here]}$$

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Helpful for building geometric intuition: ( ) moves in the direction of the steepest descent perpendicular to level lines ( ) = towards ever smaller values of .

The term $|\ |^2$ **2** represents the globally confining parabolic potential a deep well on the surface of ( ). The random potential ( ) generates many local minima of ( ) shallow wells . Have two competing terms...

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Consider **non-linear** systems $\dot{} = - \quad + {}_{,}(\quad) \quad \in \quad {}^{N}$ with

$${}_{i}(\quad) = -$$

A signature of the May-Wigner transition on the global scale

Let $\mathcal{N}_{tot}$ be the total number of equilibrium pnts of $\dot{} = - \quad + , (\quad)$.

These are solutions of the system of equations $- \quad + , (\quad) = \quad .$

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Introduce dimensionless $= \frac{}{2\,\sqrt{}}$ where $= \sqrt{\ ^2 + \ ^2}$ interaction strength.

Theorem. **[YF and BK 2016] Assume $0 \leq \ \ 1$. To leading order for large,**

$$\langle \mathcal{N}_{tot} \rangle = \begin{cases} 1 & \text{if } \quad 1 \\ \sqrt{\frac{2(1+\ )}{1-}} \; N\,\Sigma_{tot}(m) & \text{if } 0 \quad 1 \end{cases}$$

where $\Sigma_{tot}(\ ) = \frac{m^2-1}{2} - \ln$. Moreover, the relative width of the crossover region is $\ ^{-1/2}$ and the crossover profile of $\langle \mathcal{N}_{tot} \rangle$ can be found in closed form.

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Rice-Kac and reduction to RMT

Want to count zeros of $-$ $+$ $,$ $($ $)$. By Kac-Rice

$$\mathcal{N}_{tot} = \int_{\mathbb{R}^N} (- \quad + \, (\;)) \left| \det \left(- \quad {}_{ij} + \frac{{}_i}{{}_j} (\;) \right) \right| \quad ,$$

Homogeneity and Gaussianity imply indep ce of $,$ $(\,)$ and $\frac{f_i}{x_j}(\,)$ hence

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$$\langle \mathcal{N}_{tot} \rangle = \frac{1}{N} \langle |\det(- \quad + \quad)| \rangle \qquad = \left(\frac{{}_i}{{}_j} \right).$$

Matrix is Gaussian with zero mean and matrix entry correlators

$$\langle {}_{ij} \, {}_{nm} \rangle = {}^2({}_{in} \, {}_{jm} + {}_{jn} \, {}_{im} + {}_{ij} \, {}_{nm}) + (1 \quad).$$

Thus $= (X + \sqrt{\quad} \;)$ where $X \sim$ RealGin( ) and $\sim$ (0 1) independent.

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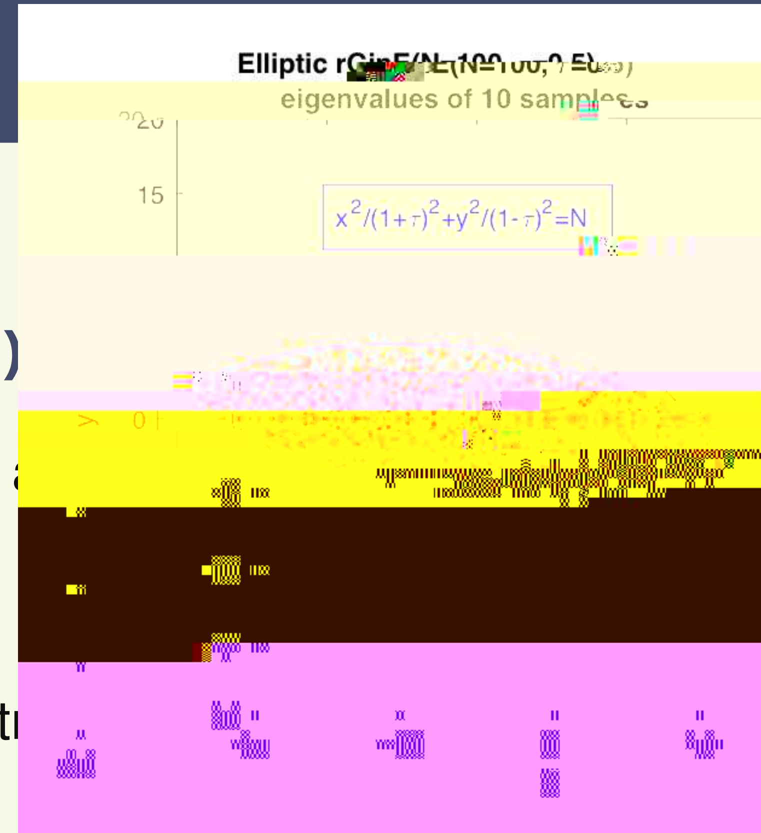

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Thus $= (X + \sqrt{\ }\ )$ where $X \sim$ RealGin( ) and $\sim$ (0 1) independent.

$$\therefore \quad \langle\mathcal{N}_{tot}\rangle = \frac{1}{N\ \ N/2}\int_{-\infty}^{\infty}\langle|\det[\ \ - X]|\rangle_X \frac{-\frac{Nt^2}{2}}{\sqrt{2\pi}}$$

where $= (\ \ + \ \sqrt{\ })\sqrt{\ }$ and $(X) \propto \exp\left[-\frac{1}{2(1-\ ^2)}\left(\mathrm{Tr}\, XX^T - \ \mathrm{Tr}\, X^2\right)\right]$

Analytic problem: find the average of the abs value of the characteristic polynomial in the real elliptic Ginibre ensemble.

Edelman-Kostlan-Shub trick

Start with the real elliptic ensemble X_{N+1} of **(+ 1)** $\times$ **(+ 1)** matrices.

Decompose X_{N+1} = , $\begin{pmatrix} & \\ \mathbf{0} & X_N \end{pmatrix}$, ^{T} where is a real eigenvalue of X_{N+1} and , is an orthogonal matrix that exchanges the corresponding eigenvector and **(1 0** . . . **0)** Householder reflection .

The Jacobian of changing from X_{N+1} to X_N , is $|$ **det(** $_N - X_N$**)**$|$.

Note: **Tr** $X_{N+1}X_{N+1}^T$ **=** ^{2} **+** ^{T} **+ Tr** $X_N X_N^T$ and **Tr** X_{N+1}^2 **=** ^{2} **+ Tr** X_N^2.

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How many equilibria are stable?

Averaged number of stable equilibria $\langle \mathcal{N}_{st} \rangle$ via Rice-Kac:

$$\langle \mathcal{N}_{st} \rangle = \frac{1}{N \quad N/2} \int_{-\infty}^{\infty} \langle \det(\quad , \quad - X) \quad {}_x(X) \rangle_X \frac{\quad^{-\frac{Nt^2}{2}} ,}{\sqrt{2\pi}}$$

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where $= (\quad + \sqrt{\quad})\sqrt{\quad}$ and $_x(X) = 1$ if all EVs X have real parts less than and $_x(X) = 0$ otherwise. No need for absolute value because of

For pure gradient.fields the integrand can be related to the pdf of the maximal EV of the GOE matrix Fyodorov & Nadal 2012 Auffinger Ben Arous & Cerny 2013.

This yields $\langle \mathcal{N}_{st} \rangle \to 1$ if $\quad 1$ and if $0 \quad 1$ then to leading order in $\langle \mathcal{N}_{st} \rangle \propto \; ^{N\Sigma_{st}}$ with $0 \quad \Sigma_{st} \quad \Sigma_{tot}$ Fyodorov & Nadal 2012.

Thus for purely gradient dynamics as the complexity increases there is an abrupt change from a simple set of equilibria typically a single stable equilibrium to a phase portrait dominated by an exponential number of unstable equilibria with an admixture of a smaller but still exp in number of stable equilibria.

Bouchaud s conjecture: in the general case of non-gradient dynamics there

Bouchaud's conjecture verified

Claim Ben Arous Fyodorov Kh unpublished work in progress : For 0

Conclusion

- A simple model for generic large complex systems is introduced and the dependence of the total number of equilibria on the system complexity as measured by the number of d.f. and the interaction strength is examined.

- Our outlook is complementary to that of May's in that it adopts a global point of view which is not limited to the neighbourhood of the presumed equilibrium.

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- A simple model for generic large complex systems is introduced and the dependence of the total number of equilibria on the system complexity as measured by the number of d.f. and the interaction strength is examined.
- Our outlook is complementary to that of May's in that it adopts a global point of view which is not limited to the neighbourhood of the presumed equilibrium.
- Our main finding is that in the presence of interactions as the complexity increases there is an abrupt change from a simple set of equilibria (typically

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Open Problems:

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- Universality of the emerging picture? How to extend the calculations beyond homogeneous Gaussian fields?
- Completely open problem: global dynamical behaviour for a generic non-potential random flow existence and stability of limit cycles emergence of chaotic dynamics etc.

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