

Polynomial Ensembles of Derivative Type

Holger Kösters

(joint work with Mario Kieburg)

Bielefeld University

Overview



Overview

1 Motivation

2 Polynomial Ensembles . 0.606.616 cm Q n Q 1 - 0.98032 0.98047 0.9

Motivation

Standing Assumption:

Let \mathbf{X}_n be a *bi-invariant* (*bi-unitarily invariant*) (*isotropic*) random matrix with values in $G := \text{GL}(n; \mathbb{C})$.

\mathbf{X}_n bi-invariant \implies for any $V, W \in K := \text{U}$

Motivation

Standing Assumption:

Let \mathbf{X}_n be a *bi-invariant* (*bi-unitarily invariant*) (*isotropic*) random matrix with values in $G := \text{GL}(n; \mathbb{C})$.

$$\mathbf{X}_n \text{ bi-invariant } \implies \text{ for any } V, W \in K := \text{U}(n), \quad V\mathbf{X}_n W \stackrel{d}{=} \mathbf{X}_n$$

Associated Densities:

- (bi-invariant) matrix density: $f_G(g)$ on $\text{GL}(n; \mathbb{C})$
jpdf of the matrix entries w.r.t. *Lebesgue measure* on $\mathbb{C}^{n \times n}$
- (symmetric) singular value density: $f_{\text{SV}}(a) = f_{\text{SV}}(a_1; \dots; a_n)$ on \mathbb{R}_+^n
jpdf of the *squared* singular values
- (symmetric) eigenvalue density: $f_{\text{EV}}(z) = f_{\text{EV}}(z_1; \dots; z_n)$ on \mathbb{C}^n
jpdf of the eigenvalues

Basic Question:

What can we say about these densities (and the relation between them)?

Ginibre Matrix

$$f_{SV}(a) \propto \prod_{j=1}^n e^{-a_j}$$
$$f_{EV}(z) \propto \prod_{j=1}^n e^{-jzj^2}$$

Fisher (1939), Hsu (1939), Roy (1939), ...

Ginibre (1965)

Motivation

Ginibre Matrix

$$f_{SV}(a) \propto \int \prod_{j=1}^n |a_j|^2 e^{-|a_j|^2}$$

Fisher (1939), Hsu (1939), Roy (1939), ...

$$f_{EV}(z) \propto \int \prod_{j=1}^n |z_j|^2 e^{-|z_j|^2}$$

Ginibre (1965)

Product of p Ginibre Matrices

$$f_{SV}(a) \propto \int \prod_{j=1}^n |a_j|^{2p} \det \left(|a_k|^{2p} \right)^{j-1} w_{p,0}(a_k)$$

Akemann{Kieburg{Wei (2013)

$$f_{EV}(z) \propto \int \prod_{j=1}^n |z_j|^{2p} w_{p,0}(|z_j|^2)$$

Akemann{Burda (2012)

where $w_{p,q}(x) = G_{q;p}^{p;q} \begin{matrix} n, \dots, n \\ 0, \dots, 0 \end{matrix} x = \frac{1}{2} \int_{-i}^{1+i} \Gamma(1+i\tau) \Gamma(1-i\tau) \Gamma(p) \Gamma(q) (1+n-s) x^{-s} ds$

Motivation

Ginibre Matrix

$$f_{SV}(a) \propto \prod_{j=1}^n n(a) j^2 e^{-a_j}$$
$$f_{EV}(z) \propto \prod_{j=1}^n n(z) j^2 e^{-jzj^2}$$

Fisher (1939), Hsu (1939), Roy (1939), ...

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Motivation

Common Structure:

$$f_{SV}(a) \propto \int_n(a) \det \left(a_k \frac{\partial}{\partial a_k} \right)^{j-1} w(a_k)$$
$$f_{EV}(z) \propto \int_n(z) j^2 \prod_{j=1}^n w(|z_j|^2)$$

Several Examples:

- products of independent [inverse] Ginibre matrices
- products of independent [inverse] truncated unitary matrices
- mixed products

Akemann{Burda (2012), Akemann{Strahov (2013), Akemann{Kieburg{Wei (2013), Akemann{Ipsen{Kieburg (2013), Adhikari{Reddy{Reddy{Saha (2013), Ipsen{Kieburg (2014), Akemann{Burda{Kieburg{Nagao (2014), Forrester (2014), Akemann{Ipsen{Strahov (2014), Kuijlaars{Zhang (2014), Kuijlaars{Stivigny (2014), Kieburg{Kuijlaars{Stivigny (2015), Kuijlaars (2015), Akemann{Ipsen (2015), Claeys{Kuijlaars{Wang (2015), ...

Overview

Polynomial Ensembles of Derivative Type

Polynomial Ensemble

Kuijlaars{Stivigny (2014)

X_n is from a *polynomial ensemble* if it is bi-invariant and

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Polynomial Ensembles of Derivative Type

Polynomial Ensemble Kuijlaars{Stivigny (2014)}

\mathbf{X}_n is from a *polynomial ensemble* if it is bi-invariant and

$$f_{SV}(a) \propto \prod_{j,k=1,\dots,n} w_j(a_k)$$

for some weight functions w_1, \dots, w_n (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$.

Theorem (Transfer Law) Kuijlaars{Stivigny (2014)}

If $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim \text{Ginibre}$ are independent, then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \sim w_{\text{Gin}}, \dots, w_n \sim w_{\text{Gin}})$, where $w_{\text{Gin}}(x) := e^{-x^2}$.

Kuijlaars{Stivigny (2014), Kieburg{Kuijlaars{Stivigny (2015), Kuijlaars (2015), Claeys{Kuijlaars{Wang (2015), ...

Polynomial Ensemble of Derivative Type Kieburg{K. (2016)}

\mathbf{X}_n is from a *polynomial ensemble of derivative type* if it is bi-invariant and

$$f_{SV}(a) \propto \prod_{j,k=1,\dots,n} \left(a_k \frac{\partial}{\partial a_k} \right)^{j-1} w_0(a_k)$$

for some weight function w_0 (with suitable properties).

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Polynomial Ensembles of Derivative Type

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\mathbf{X}_n is from a *polynomial ensemble of derivative type* if it is bi-invariant and

$$f_{SV}(a) \propto \int_n(a) \det \left(a_k \frac{\partial}{\partial a_k} \right)^{j-1} w_0(a_k) \quad j,k=1,\dots,n$$

for some weight function w_0 (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim DPE(w_0)$.

Examples from RMT

- induced Wishart{Laguerre ensemble:
- induced Jacobi ensemble:
- induced Cauchy{Lorentz ensemble:

- products of such random matrices:

$$w_0(a) = a e^{-a}$$

$$w_0(a) = a (1-a)^{j-1} \mathbf{1}_{(0,1)}(a)$$

$$w_0(a) = a (1+a)^{-j-1}$$

$$w_0(a) = \text{Meijer-G-function}$$

- Muttalib{Borodin ensemble (of Wishart{Laguerre type)

Muttalib (1995), Borodin (1999), Cheliotis (2014), Forrester{Liu (2014), Forrester{Wang (2015), Zhang (2015), ...

$$(a) f_{SV}(a) \propto \int_n(a) \det(a) e^{-\text{tr} a}$$

$$w_0(a) = a e^{-a}$$

$$(b) f_{SV}(a) \propto \int_n(a) \det(a) e^{-\text{tr}(\ln a)^2}$$

$$w_0(a) = a e^{-(\ln a)^2}$$



Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{SV}(a) \propto \frac{1}{n} \det \left(\frac{a_k}{a_k} \right)_{j,k=1,\dots,n}^{-1} w_0(a_k)$.

Theorem (Eigenvalue Density) Kieburg (K. (2016))

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{EV}(z) \propto \int \prod_{j=1}^n w_0(jz_j^2) dz_j$.

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{SV}(a) \propto \frac{1}{n(a)} \det \left(a_k \frac{\partial}{\partial a_k} \right)^{j-1} w_0(a_k)_{j,k=1,\dots,n}$.

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Theorem (Transfer Law) Kieburg{K. (2016)}

Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \sim v_0)$, where

$$(w_0 \sim v_0)(x) = \int_0^x w_0(xy^{-1}) v_0(y) \frac{dy}{y}; \quad x > 0:$$

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More generally, let $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \sim v_0, \dots, w_n \sim v_0)$.

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{SV}(a) \propto \prod_{k=1, \dots, n} (a_k \frac{a}{a_k})^{j-1} w_0(a_k)$.

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Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \sim v_0, \dots, w_n \sim v_0)$.

Theorem (Transfer Law) Kieburg{K. (2016)}

Let $\mathbf{X}_n \sim DPE(w_0)$. Then $\mathbf{X}_n^1 \sim DPE(w_0)$, where $w_0(x) = w_0(x^{-1}) x^{n-1}$.

Key Tool: Spherical Transform

Univariate Situation

X : r.v. with values in \mathbb{C}
and a **rotation-invariant** density f_X

dist. of X ! dist. of jXj^2

jXj^2 : r.v. with values in \mathbb{R}_+
and a density f_{jXj^2}

Mellin Transform

$$M_X(s) = \int_{\mathbb{C}} f_X(x) jxj^{2s} \frac{dx}{jxj^2}$$

$$= \int_{\mathbb{R}_+} f_{jXj^2}(y) y^s \frac{dy}{y}$$

for suitable $s \in \mathbb{C}$

Uniqueness Theorem

$$M_{X_1} = M_{X_2} \Rightarrow X_1 \stackrel{d}{=} X_2$$

Multiplication Theorem

$$X_1; X_2 \text{ ind.} \Rightarrow M_{X_1 X_2} = M_{X_1} M_{X_2}$$

Multivariate Situation

\mathbf{X} : r.v. with values in $GL(n; \mathbb{C})$
and a **bi-invariant** density $f_{\mathbf{X}}$

dist. of \mathbf{X} ! dist. of $\mathbf{X} \mathbf{X}$

$\mathbf{X} \mathbf{X}$: r.v. with values in $Pos(n; \mathbb{C})$
and a **conjugation-invariant** density $f_{\mathbf{X} \mathbf{X}}$

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n; \mathbb{C})} f_{\mathbf{X}}(\mathbf{x}) \det(\mathbf{x})^{-s} \frac{d\mathbf{x}}{j\det \mathbf{x}j^{2n}}$$

$$= \int_{Pos(n; \mathbb{C})} f_{\mathbf{X} \mathbf{X}}(y) \det(y)^{-s} \frac{dy}{(\det y)^n}$$

for suitable $s \in \mathbb{C}^n$

Uniqueness Theorem

$$S_{\mathbf{X}_1} = S_{\mathbf{X}_2} \Rightarrow \mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$$

Multiplication Theorem

$$\mathbf{X}_1; \mathbf{X}_2 \text{ ind.} \Rightarrow S_{\mathbf{X}_1 \mathbf{X}_2} = S_{\mathbf{X}_1} S_{\mathbf{X}_2}$$

$\int_{\mathbb{C}^n} f(\mathbf{x}) \det(\mathbf{x})^{-s} \frac{d\mathbf{x}}{j\det \mathbf{x}j^{2n}}$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n;\mathbb{C})} f_{\mathbf{X}}(x) |s(x)| \frac{dx}{|\det x|^{2n}} \quad (s \in \mathbb{C}^n)$$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{\text{GL}(n; \mathbb{C})} f_{\mathbf{X}}(x) \cdot s(x \cdot x) \frac{dx}{j \det x^{2n}} \quad (s \in \mathbb{C}^n)$$

Spherical Function

Let $x \in \text{GL}(n; \mathbb{C})$, let $\mathbf{X}_n \in \text{GL}(n; \mathbb{C})$ be a bi-invariant random matrix with the same singular values as x , and let $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$ be its *QR decomposition*.

$$\cdot s(x \cdot x) := \mathbb{E} \prod_{j=1}^n R_{jj}^{2(s_j + \frac{1}{2})} \quad (\text{where } \frac{1}{2} := j - \frac{n+1}{2})$$

Key Tool: Spherical Transform

Spherical Transform

$$S_{\mathbf{X}}(s) = \int_{GL(n; \mathbb{C})} f_{\mathbf{X}}(x) ' s(x x) \frac{dx}{j \det x j^{2n}} \quad (s \in \mathbb{C}^n)$$

Spherical Function

Let $x \in GL(n; \mathbb{C})$, let $\mathbf{X}_n \in GL(n; \mathbb{C})$ be a bi-invariant random matrix with the same singular values as x , and let $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$ be its *QR decomposition*.

$$' s(x x) := \mathbb{E} \prod_{j=1}^n R_{jj}^{2(s_j + \frac{1}{2} j - 1)} \quad (\text{where } \frac{1}{2} j := j - \frac{1}{2} n)$$

Some Ideas from the Proofs

Proposition (Spherical Transform) Kieburg{K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $S_{\mathbf{X}_n}(s) \propto \prod_{k=1}^n (M_{w_0})(s_k - \frac{n-1}{2}) :$

Some Ideas from the Proofs

Proposition (Spherical Transform) Kieburg{K. (2016)}

If $\mathbf{X}_n \sim DPE(w_0)$, then $S_{\mathbf{X}_n}(s) \propto \prod_{k=1}^n (Mw_0)(s_k - \frac{n-1}{2})$:

Proof:

Set $Df(x) := (x) f^0(x)$ and note that $M(Df)(s) = s Mf(s)$.

$$\begin{aligned}
 S_{\mathbf{X}}(s) &= \int_{\mathbb{Z}} \int_{\text{GL}(n; \mathbb{C})} f_{\mathbf{X}}(x) \cdot s(x, x) \frac{dx}{j \det x^{2n}} = \int_{(0; 1)^n} f_{SV}(\cdot) \cdot s(\cdot) \frac{d}{(\det \cdot)^n} \\
 &\propto \int_{(0; 1)^n} n(\cdot) \det D^j \cdot^{-1} w_0(\cdot) \frac{\det(\cdot)_j^{s_k + (n-1)/2}}{n(s) \cdot n(\cdot)} \frac{d}{(\det \cdot)^n} \\
 &\propto \frac{1}{n(s)} \det \int_0^1 D^j \cdot^{-1} w_0(\cdot) s_k \cdot^{(n+1)/2} d \\
 &= \frac{\det(s_k - \frac{n-1}{2})^j \cdot^{-1} (Mw_0)(s_k - \frac{n-1}{2})}{n(s)} = \prod_{k=1}^n (Mw_0)(s_k - \frac{n-1}{2}) :
 \end{aligned}$$

Some Ideas from the Proofs

Theorem (Transfer Law for the Product) Kieburg{K. (2016)

Let $\mathbf{X}_n \stackrel{\text{DPE}}{\sim} (w_0)$ and $\mathbf{Y}_n \stackrel{\text{DPE}}{\sim} (v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \stackrel{\text{DPE}}{\sim} (w_0 \sim v_0)$, where $(w_0 \sim v_0)(x) = \int_0^x w_0(xy^{-1}) v_0(y) \frac{dy}{y}$.

Proof:

$$\mathbf{X}_n \stackrel{\text{DPE}}{\sim} (w_0); \mathbf{Y}_n \stackrel{\text{DPE}}{\sim} (v_0)$$

$$\Rightarrow S_{\mathbf{X}_n}(s) \stackrel{\text{DPE}}{\sim} M_{w_0}(s_k \frac{n-1}{2}); S_{\mathbf{Y}_n}(s) \stackrel{\text{DPE}}{\sim} M_{v_0}(s_k \frac{n-1}{2})$$

$$\Rightarrow S_{\mathbf{X}_n \mathbf{Y}_n}(s) \stackrel{\text{DPE}}{\sim} M(w_0 \sim v_0)(s_k \frac{n-1}{2})$$

$$\Rightarrow \mathbf{X}_n \mathbf{Y}_n \stackrel{\text{DPE}}{\sim} (w_0 \sim v_0)$$

Some Ideas from the Proofs

Theorem (Eigenvalue Density)

Kieburg{K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{EV}(z) \sim \prod_{j=1}^n (z_j)^2 \int_T^Z w_0(jz_j)^2 dt$.

Proof:

$$f_{EV}(z) \sim \prod_{j=1}^n (z_j)^2 \int_T^Z \prod_{j=1}^n (jz_j)^{2n-2j} f_G(zt) dt$$

$$f_{(\mathbf{R}_{11}, \dots, \mathbf{R}_{nn})}(r_1, \dots, r_n) \sim \prod_{j=1}^n (r_j)^{2n-2j} \int_T^Z f_G(rt) dt$$

Some Ideas from the Proofs

Theorem (Eigenvalue Density)

Kieburg{K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{EV}(z) \sim \prod_{j=1}^n (z_j)^2 w_0(jz_j^2)$.

Proof:

$$f_{EV}(z) \sim \prod_{j=1}^n (z_j)^2 \int_{\mathcal{Z}} \prod_{j=1}^n jz_j^{2n-2j} f_G(zt) dt$$

$$f_{(\mathbf{R}_{11}, \dots, \mathbf{R}_{nn})}(r_1, \dots, r_n) \sim \prod_{j=1}^n jz_j^{2n-2j} \int_{\mathcal{T}} f_G(rt) dt$$

Thus, since $S_{\mathbf{X}}$ is essentially the (componentwise) Mellin transform of $f_{(\mathbf{R}_{11}, \dots, \mathbf{R}_{nn})}$, we get $f_{(\mathbf{R}_{11}, \dots, \mathbf{R}_{nn})}$, and hence f_{EV} , from $S_{\mathbf{X}}$ by (componentwise) Mellin inversion:

$$S_{\mathbf{X}_n}(s) \sim \prod_{j=1}^n M w_0(s_j - \frac{n-1}{2}) \quad f_{EV}(z) \sim \prod_{j=1}^n (z_j)^2 w_0(jz_j^2)$$

Overview



Interpolation between Product Ensembles I

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{SV}^{(p;q)}(a) \propto \frac{1}{n(a)} \det \left(a_k \frac{a_l}{a_k} \right)^{j-1} w_{p;q}(a_k)$$

$$f_{EV}^{(p;q)}(z) \propto \int \frac{1}{j} n(z) j^2 \prod_{j=1}^n w_{p;q}(j z_j j^2)$$

$$\text{where } w_{p;q}(x) = \frac{1}{2} \int_{-i}^{1+i} x^{s-1} \Gamma(1+i-s) \Gamma(1+n-s) x^{-s} ds$$

Interpolation between Product Ensembles I

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{SV}^{(p;q)}(a) \propto n(a) \det \left(a_k \frac{a^k}{a_k} \right)^{j-1} w_{p;q}(a_k)$$

$$f_{EV}^{(p;q)}(z) \propto j \cdot n(z) j^2 \prod_{j=1}^n w_{p;q}(jz_j^2)$$

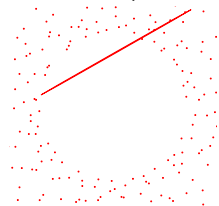
$$\text{where } w_{p;q}(x) = \frac{1}{2} \int_{-1}^{1+i\epsilon} \int_{-1-i\epsilon}^{1+i\epsilon} x^s ds \int_{-1-i\epsilon}^{1+i\epsilon} p(s) q(1+n-s) x^s ds$$

Problem: Is it possible to interpolate between these "product ensembles"?

Motivation: *heavy-tailed* one-point densities (in the limit as $n \rightarrow \infty$)

$$g_{SV}^{(p;q)}(a) \sim a^{\frac{q+3}{q+1}} \quad (a \rightarrow \infty)$$

$$g_{EV}^{(p;q)}(z) \sim |z| j^{\frac{2q+2}{q}} \quad (z \rightarrow \infty)$$



Interpolation between Product Ensembles II

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{SV}^{(p;q)}(a) \propto \frac{1}{n(a)} \det \left(a_k \frac{\partial}{\partial a_k} \right)^{j-1} w_{p;q}(a_k)$$

$$f_{EV}^{(p;q)}(z) \propto \int \frac{1}{j} n(z) j^2 \prod_{j=1}^n w_{p;q}(jz_j j^2)$$

$$\text{where } w_{p;q}(x) = \frac{1}{2} \int_{-i}^{i} x^{1+i} \Gamma(1+i) \Gamma(1-i) \Gamma(p(s) + q(1+n-s)) x^{-s} ds$$

Connection to Polya Frequency Functions

Polya Frequency Function

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called *Polya frequency function* of order n (PF_n) if

$$\det(f(x_j - y_k))_{j,k=1,\dots,m} > 0$$

for any $m = 1, \dots, n$ and any $x_1 < \dots < x_m, y_1 < \dots < y_m$.

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for any $m = 1, \dots, n$ and any $x_1 < \dots < x_m, y_1 < \dots < y_m$.

Proposition Kieburg (K. (2016))

If $f \in \text{PF}_n$ (with suitable differentiability and integrability properties), then $w_0 := f \circ \log$ gives rise to a random matrix $\mathbf{X}_n \sim \text{DPE}(w_0)$.

Summary and Outlook

Polynomial Ensembles of Derivative Type

- singular value and eigenvalue densities with special *determinantal* structure
- special structure is preserved under independent products (transfer laws)
- key tool: spherical transform
- many examples (from RMT and via Polya frequency functions)

Summary and Outlook

Polynomial Ensembles of Derivative Type

- singular value and eigenvalue densities with special *determinantal* structure
- special structure is preserved under independent products (transfer laws)
- key tool: **spherical transform**
- many examples (from RMT and via Polya frequency functions)

Open Problems (partly work in progress)

- which functions w_0 define polynomial ensembles of derivative type?
- limiting spectral distributions at the *global* and *local* level?
- closer connection to free probability (refined convergence results?)
- power-law decay: new applications of random matrix theory?
- extension to real and quaternionic matrices?

