Eigenvalues statistics for the GUE

Transition at mesoscopic scale

Free fermions at positive temperature

The joint density of N fermions at temperature $T > 0$ is given by

$$p_{N,T}(x_1,\dots,x_N) = \frac{1}{Z_N(T)N!} \sum_{n_1<\cdots<n_N} \det\left(\varphi_{n_i}(x_j)\right)_{i,j=1,\dots,N}^2 \exp\left(-\frac{1}{T}\sum_{i=1}^{N} \varepsilon_{n_i}\right).$$

This p.d.f. does not define a determinantal point process. However, it is known that the corresponding grand-canonical ensemble is determinantal with correlation kernel

$$K_T^N(x,y) = \sum_{n=0}^{\infty} \frac{1}{e^{(\varepsilon_n - \mu)/T}+1}\varphi_n(x)\varphi_n(y).$$

Cumulants method

if Z is a real-valued random variable, we define its cumulants $C^n[Z]$ by

$$_Z(t) = \log E\ e^{tZ} = \sum_{n=1}^{\infty} C^n[Z]\frac{t^n}{n!}$$

If is a determinantal process on $(X;)$ with correlation kernel K, then for any function $g \in L^1(X \to R)$ with compact support in A,

$$E \prod_{\in} 1 + g( ) = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A^k} \prod_{i=1}^{k} g(x_i) \det_{k\ k} (K(x_i; x_j))\, d^{\,k}(x)$$
$$= \det\ I + gK_{L^2(A;)}:$$

Taking $g( ) = e^{f( )} 1$, we obtain an explicit formula for the cumulant generating function of the linear statistics $f() := \sum_{\in} f( )$.

Cumulants method

If the kernelK is reproducing, i.e. $K(x;y)K(y;z)d\ (y) = K(x;z)$, then we obtain

$$C^n\ (f) = \int_{x_0=x_n} {}^n_0[f](x) \prod_{1\ j\ n} K(x_j;x_{j\ 1})d\ ^n(x):$$

where for all$x\ 2\ X^n$,

$$ {}^n_0[f](x) = \sum_{k_1+\ \ +k_` = n} \frac{(\ \ 1)^`}{`} \frac{n!}{k_1!\ \ \ k_`!} \prod_{1\ j\ `} f(x_j)^{k_j}:$$

Then, to prove that

$$_N(f)\ \ E[\ _N(f)]\ )\ N\ \ 0; kf k^2_H\ \ \ \text{as} N\ !\ 1\ \ ;$$

it su ces to show that for there exists$m >\ 2$ so that

$$\lim_{N!1}\ C^n\ \sum_{2\ \ N} f\ ($$ >48.25 /F11 34F34 6.97327854d [(it)-333(suF15 9.962f 2.614 0 Td |

-ensembles at = 2

Consider the Hamiltonian on X^N

$$H_V^N(x) = \sum_{1 \quad i<j \quad N} \log |x_i \quad x_j|^{\quad 1} + N \sum_{1 \quad i \quad N} V(x_i)$$

and the Boltzmann-Gibbs measure with density $_N(x) \propto e^{\quad HHk) \neq 8 \; 1.778 \; 11. \; minant6 \; 0}$

The critical regime

Theorem

Suppose that the potential V is real analytic on $\mathbb{R}^d$. Let $f \in C_c^3(\mathbb{R}^d)$ and $0 < \ < 1$. If $\lim_{N\to\infty} qN^{1} = \ $, then

$$\lim_{N\to\infty} \mathbb{E}\left[e^{\hat{X}_N(f)} \right] = \exp\left(\|f\|^2_{H^{d/2}(\mathbb{R}^d)} + \varrho_{eq}(x_0) \int_{\mathbb{R}^d} e^{f(x)} \ 1 \ f(x) \, dx \right).$$